Nonlinear Analysis 71 (2009) 2094–2113

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

ULM'S method without derivatives

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ARTICLE INFO

Article history:
Received 7 August 2007
Accepted 12 January 2009

MSC:
47J25
65J15

Keywords:
Nonlinear operator equations
Iterative methods
Convergence analysis
Functional equations

ABSTRACT

We consider a version of Ulm's method obtained when the derivative in its definition is replaced by the divided difference operator. The convergence analysis of the proposed method, carried out under the *regular continuity* assumption, more general and more flexible than the traditional Lipschitz continuity, has produced the convergence condition, existence and uniqueness radii, and error bounds, all shown to be sharp.

1. Introduction

In [1] Ulm proposed to solve nonlinear operator equations

$$\mathbf{f}(x) = 0, \qquad \mathbf{f} : X \supset D \to W, \quad X, W \text{ are Banach spaces,} \tag{1.1}$$

by the following iterative method:

$$\mathbf{B}_{n+1} := 2\mathbf{B}_n - \mathbf{B}_n \mathbf{f}'(x_n)\mathbf{B}_n, \qquad x_{n+1} := x_n - \mathbf{B}_{n+1}\mathbf{f}(x_n),$$

(see also [2] for its analysis and a numerical example). The method has several attractive properties. First, it is (like Newton's method) self-correcting. Second, it is able to converge with Newton-like rate. Third, it is (unlike Newton's method) "inversion free": no linear problem has to be solved at each iteration. Fourth, apart from solving the problem (1.1), the method generates successive approximations $\mathbf{B}_n$ to the inverse derivative $\mathbf{f}'(x_\infty)^{-1}$ at the solution x_∞, which is very helpful when one is interested in the solution's sensitivity to small perturbations. So, in fact, the method solves the system

$$\mathbf{f}(x) = 0 \quad \& \quad \mathbf{X}\mathbf{f}'(x) = \mathbf{I}$$

for the pair $(x, \mathbf{X})$, $x \in D \subset X$, $\mathbf{X} \in \mathcal{L}(W, X)$ (the space of bounded linear operators acting from W into X). At the same time, the method has a serious shortcoming: the derivative $\mathbf{f}'(x)$ has to be evaluated at each iteration. This makes it unapplicable to equations with nondifferentiable operators and in situations when evaluation of the derivative is too costly.

The secant method

$$x_{n+1} := x_n - [x_n, x_{n-1}, \mathbf{f}]^{-1}\mathbf{f}(x)$$

is free from this shortcoming, but suffers from another one: it is not inversion free. So, it is only natural to try to marry these two interesting methods to obtain a new one, which would retain valuable traits of both, but none of their undesirable ones.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.01.044

This idea is not new. It is suggested in a certain form already in [1] and rediscovered in [3]. Later the hybrid method

$$\mathbf{B}_{n+1} := 2\mathbf{B}_n - \mathbf{B}_n[x_n, x_{n-1}, \mathbf{f}]\mathbf{B}_n, \qquad x_{n+1} := x_n - \mathbf{B}_{n+1}\mathbf{f}(x_n), \tag{1.2}$$

was discussed in [4,5] (I am unaware of more recent publications where this method was mentioned). In those works, the method (1.2) is studied under assumption that the divided difference operator $[x, y, \mathbf{f}]$ is Lipschitz continuous in the sense that

$$\max\{\|[x, y, \mathbf{f}] - \mathbf{f}'(x)\|, \|[x, y, \mathbf{f}] - \mathbf{f}'(y)\|\} \le c\|x - y\|. \tag{1.3}$$

The purpose of the present paper is to offer a new Kantorovich-type convergence analysis of the hybrid method (1.2) based on a more general and more flexible continuity assumption which we call *regular continuity* [6,7]. To make the paper self-contained, we recall briefly in the next section some facts from [8] pertinent to this notion that are necessary for understanding subsequent developments. Also, two new examples are provided there, demonstrating techniques for finding a regular continuity modulus of a divided difference of a nonlinear operator. A majorant generator for (1.2) is devised and discussed in Section 3. The convergence theorem is proved and commented upon in Section 4.

2. Regularly continuous divided differences

Let $\mathbf{f}$ be a nonlinear operator acting from an open convex subset D of a Banach space X into another Banach space W.

Definition 2.1. A linear bounded operator $\mathbf{A}$ from X into W is called a divided difference operator (briefly dd), if for any given pair (x, y) of points of D it satisfies the (secant) equation

$$\mathbf{A}(x - y) = \mathbf{f}(x) - \mathbf{f}(y).$$

To emphasize its dependence on x, y, and $\mathbf{f}$, such an operator is designated by the symbol $[x, y, \mathbf{f}]$.

For given $x \in X$ and $w \in W$, linear operators satisfying the equation $\mathbf{A}x = w$ constitute an affine manifold in the space $\mathcal{L}(X, W)$ of all bounded linear operators between X and W:

$$\mathbf{A}_0 x = w \quad \& \quad \mathbf{A}x = w \Longrightarrow (\mathbf{A} - \mathbf{A}_0)x = 0 \Longrightarrow \mathbf{A} \in \mathbf{A}_0 + \mathcal{L}_x,$$

where $\mathcal{L}_x \subset \mathcal{L}(X, W)$ is the subspace of operators vanishing on x. So, the symbol $[x, y, \mathbf{f}]$ should be understood as the notation for this manifold or, more precisely, as its particular representative selected from it according to a certain rule specified in advance. If $[x, y, \mathbf{f}]$ is selected to be continuous at x with respect to y, then it is easy to see that $[x, x, \mathbf{f}] = \mathbf{f}'(x)$. Otherwise, $\lim_{t\to+0}[x, x + th, \mathbf{f}]h = \lim_{t\to+0} t^{-1}[\mathbf{f}(x + th) - \mathbf{f}(x)]$ (if it exists) may vary depending on h, $\|h\| = 1$. In this case, this limit is the directional derivative $\mathbf{f}'(x, h)$ of $\mathbf{f}$ in the direction h.

Numerous convergence analyses of the secant method appearing in literature impose on $[x, y, \mathbf{f}]$ one or another continuity assumption, more general than (1.3). For example, Potra in [9,10] assumes dd to be a *consistent approximation* to the derivative:

$$\|[x, y, \mathbf{f}] - \mathbf{f}'(u)\| \le c(\|x - u\| + \|y - u\|), \quad \forall x, y, u \in D. \tag{2.4}$$

In [11,12], the inequality

$$\|[x, y, \mathbf{f}] - [u, v, \mathbf{f}]\| \le c(\|x - u\| + \|y - v\|), \quad \forall x, y, u, v \in D. \tag{2.5}$$

(*Lipschitz continuity* of dd) is required. In [13] Hernández and Rubio replace Lipschitz continuity by the more general *Hölder continuity*, which means that

$$\|[x, y, \mathbf{f}] - [u, v, \mathbf{f}]\| \le c(\|x - u\|^p + \|y - v\|^p), \quad \forall x, y, u, v \in D. \tag{2.6}$$

for some $p \in (0, 1]$. In [14–16] these authors relax this requirement still further, assuming that a continuous nondecreasing function $\omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ is known such that

$$\|[x, y, \mathbf{f}] - [u, v, \mathbf{f}]\| \le \omega(\|x - u\|, \|y - v\|), \quad \forall x, y, u, v \in D. \tag{2.7}$$

It is noted however that assumptions of the type of *ω-continuity*

$$\|[x, y, \mathbf{f}] - [u, v, \mathbf{f}]\| \le \omega(\|x - u\| + \|y - v\|), \quad \forall x, y, u, v \in D. \tag{2.8}$$

are too general for meaningful convergence analysis. First, we observe that the least ω satisfying (2.8)

$$\underline{\omega}(t) := \sup_{x,y,u,v} \left\{\|[x, y, \mathbf{f}] - [u, v, \mathbf{f}]\| \;\middle|\; (x, y, u, v) \in D^4 \quad \& \quad \|x - u\| + \|y - v\| \le t \right\},$$

in addition to being continuous and nondecreasing, vanishes at zero and subadditive: $\underline{\omega}(s+t) \le \underline{\omega}(s)+\underline{\omega}(t), \forall s > 0,\ t > 0$. The functions ω possessing all four properties

(i) $\omega(0) = 0$,
(ii) continuity on $[0, \infty)$,
(iii) monotonicity,
(iv) subadditivity,