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Critical exponents for porous medium systems coupled via nonlinear boundary flux ☆

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ABSTRACT

This paper is concerned with the porous medium equations for coupling via nonlinear boundary flux; we obtain the critical global existence curve and the critical Fujita curve for the problem, considered by constructing the self-similar supersolutions and subsolutions.

1. Introduction

In this paper, we consider the following Newtonian filtration system coupled via nonlinear boundary flux:

$$u_t = (u^m)_{xx}, \qquad v_t = (v^n)_{xx}, \quad (x, t) \in \mathbb{R}^+ \times [0, T), \tag{1.1}$$

$$-(u^m)_x(0, t) = u^\alpha(0, t)v^p(0, t), \qquad -(v^n)_x(0, t) = u^q(0, t)v^\beta(0, t), \quad t \in (0, T), \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^+, \tag{1.3}$$

where $0 < m, n < 1, \alpha, \beta \geq 0, p, q > 0$ and $u_0(x), v_0(x)$ are continuous, nonnegative functions with compact support.

The system (1.1)–(1.3) can be used to provide a model for nonlinear heat propagation; it also appears in several branches of applied mathematics, such as population dynamics, chemical reactions, and so on. The local existence and the comparison principle of nonnegative weak solutions can be establish using the classical theory for parabolic equations (see [1,2]).

In the present work, we are mainly interested in the long time behavior of solutions, include blow-up in a finite time and global existence in time, which has been observed for many diffusion equations with nonlinear sources since the pioneering work of Fujita [3]; for further reference, see the surveys [4,5] and the references therein. Now we recall some known results on the system (1.1)–(1.3). Galaktionov et al. [6] and Ferreira et al. [7] considered the single-equation case

$$u_t = (u^m)_{xx}, \quad (x, t) \in \mathbb{R}^+ \times (0, T), \tag{1.4}$$

$$-(u^m)_x(0, t) = u^p(0, t), \quad t \in (0, T), \tag{1.5}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^+, \tag{1.6}$$

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with $m > 1$ and $0 < m < 1$ respectively. They both proved that if $0 < p < p_0 = \frac{m+1}{2}$, then every nontrivial solution is global in time, while for $p > \frac{m+1}{2}$ there are solutions blowing up in a finite time. Thus, $p_0 = \frac{m+1}{2}$ is the critical global existence exponent for the problem (1.4)–(1.6). Moreover, they obtained that $p_c = m + 1$ is the critical Fujita exponent; by definition, p_c has the following properties:

(i) if $p_0 < p < p_c$, then the solution $u(x, t) \not\equiv 0$ of the problem (1.4)–(1.6) blows up in a finite time for all nontrivial u_0;
(ii) if $p > p_c$, then the solutions of the problem (1.4)–(1.6) are global for small u_0 and blow up in a finite time for large u_0.

For the system (1.1)–(1.3), instead of critical exponents there are critical curves, one for global existence and another of Fujita type. To state the known results on the critical curves for the system (1.1)–(1.3), we define

$$\alpha_1 = \frac{2p+n+1}{(m+1)(n+1)-4pq}, \qquad \alpha_2 = \frac{2p+m+1}{(m+1)(n+1)-4pq},$$
$$\beta_1 = \frac{p(m-1-2q)+(n+1)m}{(m+1)(n+1)-4pq}, \qquad \beta_2 = \frac{q(n-1-2p)+(m+1)n}{(m+1)(n+1)-4pq},$$
$$k_1 = \frac{2p+n+1-2\beta}{4pq-(m+1-2\alpha)(n+1-2\beta)}, \qquad l_1 = \frac{1-k_1(m-1)}{2},$$
$$k_2 = \frac{2q+m+1-2\alpha}{4pq-(m+1-2\alpha)(n+1-2\beta)}, \qquad l_2 = \frac{1-k_2(n-1)}{2}.$$

The system (1.1)–(1.3) with $\alpha = \beta = 0$ and $m > 1$ was studied by Quirós and Rossi in [8]; it was shown that the critical global existence curve is $pq = (\frac{m+1}{2})(\frac{n+1}{2})$ and the critical Fujita type curve is $\min\{\alpha_1+\beta_1, \alpha_2+\beta_2\} = 0$. In [9], Wang et al. considered the semilinear case of the system (1.1)–(1.3) with $0 < \alpha, \beta < 1$; they obtained that the critical global existence curve is $pq = (1-\alpha)(1-\beta)$. It was Zheng et al. who dealt with the general system (1.1)–(1.3) with $m > 1$ in [10], in which the authors proved that for $\alpha < \frac{m+1}{2}, \beta < \frac{n+1}{2}$, the critical global existence curve is $pq = (\frac{m+1}{2}-\alpha)(\frac{n+1}{2}-\beta)$ and the critical Fujita curve is $\min\{l_1-k_1, l_2-k_2\} = 0$, while if $\alpha > \frac{m+1}{2}$ or $\beta > \frac{n+1}{2}$, then the solutions may blow up in a finite time.

The purpose of this paper is to extend the results on critical curves for the system (1.1)–(1.3) for the slow diffusion case [10] to the fast diffusion case, namely, $0 < m < 1$. Our main results are the following theorems.

Theorem 1. *Assume $\alpha < \frac{m+1}{2}, \beta < \frac{n+1}{2}$. If $pq \le (\frac{m+1}{2}-\alpha)(\frac{n+1}{2}-\beta)$, then every solution of the system (1.1)–(1.3) exists globally in time.*

Theorem 2. *Assume $\alpha \le \frac{m+1}{2}, \beta \le \frac{n+1}{2}$. If $pq > (\frac{m+1}{2}-\alpha)(\frac{n+1}{2}-\beta)$, then the solutions of the system (1.1)–(1.3) with large initial data blow up in a finite time.*

Remark 1. From Theorems 1 and 2, we see that the critical global curve for the system (1.1)–(1.3) is $pq = (\frac{m+1}{2}-\alpha)(\frac{n+1}{2}-\beta)$ if $\alpha < \frac{m+1}{2}, \beta < \frac{n+1}{2}$.

Theorem 3. *Assume $\alpha \le \frac{m+1}{2}, \beta \le \frac{n+1}{2}, pq > (\frac{m+1}{2}-\alpha)(\frac{n+1}{2}-\beta)$.*

(i) If $l_1 < k_1$ or $l_2 < k_2$, then every nonnegative, nontrivial solution of (1.1)–(1.3) blows up in a finite time.
(ii) If $l_1 > k_1$ and $l_2 > k_2$, then the solutions of (1.1)–(1.3) are global for small initial data and blow up in a finite time for large initial data.

Remark 2. Theorem 3 indicates that the critical Fujita curve is $\min\{l_1-k_1, l_2-k_2\} = 0$ if $\alpha \le \frac{m+1}{2}, \beta \le \frac{n+1}{2}$.

For the case $\alpha > \frac{m+1}{2}$ or $\beta > \frac{n+1}{2}$, we have:

Theorem 4. *If $\alpha > \frac{m+1}{2}$ or $\beta > \frac{n+1}{2}$, then the solutions of (1.1)–(1.3) may blow up in a finite time.*

Remark 3. By Theorem 4, it is seen that the critical global curve for the system (1.1)–(1.3) is $\alpha = \frac{m+1}{2}, \beta = \frac{n+1}{2}$ if $pq = (\frac{m+1}{2}-\alpha)(\frac{n+1}{2}-\beta)$.

2. Proofs of the main results

In this section, we prove the main results of this paper. Motivated by [10,7], we introduce self-similar solutions in the following to obtain these results.