Nonlinear Analysis 71 (2009) 2291–2297

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

LaSalle's theorems in impulsive semidynamical systems

E.M. Bonotto *

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil

ARTICLE INFO

Article history:
Received 5 August 2008
Accepted 12 January 2009

MSC:
34A37
37B25
54H20

Keywords:
Impulsive semidynamical systems
Lyapunov function
Invariance principle
Asymptotic stability

ABSTRACT

We consider a semidynamical system subject to variable impulses and we obtain the LaSalle invariance principle and the asymptotic stability theorem for this semidynamical system.

1. Introduction

Impulsive differential equations (IDE) are an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. These equations are modelled by differential equations which describe the period of continuous variation of state and conditions which describe the discontinuities of first kind of the solution or of its derivatives at the moments of impulse. The theory of IDE is an important area of investigation. For details, see Refs. [1–3].

In [4], two classical LaSalle's results, namely the Invariance Principle and the Asymptotic Stability Theorem (see [5]) were proved for some very special impulsive systems. The aim of this paper is to generalize these results for more general impulsive systems.

In [4], Saroop K. Kaul considers an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, where $\Omega \subset \mathrm{X}$ is an open set in a metric space X and the continuous impulsive function I is defined from $\partial\Omega$ to X ($\partial\Omega$ is the boundary of Ω in X). The author introduces a continuous Lyapunov function in $(\Omega, \widetilde{\pi})$ given by $V : \overline{G} \to \mathbb{R}$, where $G \subset \Omega$ is a positively invariant closed set under $\widetilde{\pi}$ in Ω, $\overline{G}$ denotes the closure of G in X and V satisfies the following properties:

1. V is continuous,
2. $V(\mathrm{I}(x)) \leq V(x)$ for $x \in \overline{G} \cap \mathrm{M}$, and
3. $\dot{V}(x) \leq 0$ for $x \in G$, where $\dot{V}(x) = \lim_{t \to 0^+} \frac{V(\widetilde{\pi}(x,t)) - V(x)}{t}$.

For the continuous case (i.e. non-impulsive semidynamical systems) the reader can find further information about Lyapunov functions in [6].

Considering $E = \{x \in G : \dot{V}(x) = 0\}$ and $A \subset E$ as being the largest invariant set under $\widetilde{\pi}$, the following results are proved in [4]:

- (Invariance Principle) There exists an $\alpha \in \mathbb{R}$ such that for $x \in G$, $\widetilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\alpha)$.

* Tel.: +55 16 3373 8156; fax: +55 16 3373 9633.
E-mail address: ebonotto@icmc.usp.br.

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.01.062

- (Asymptotic Stability) If $A \subset \mathrm{int}G$, then A is an attractor. If, furthermore, $V(A) = \alpha$ for some $\alpha \in \mathbb{R}$, then A is asymptotically stable.

We can observe that the set A does not contain points of $\partial\Omega$ where the discontinuities of the impulsive system occur. Note, as well, that the limit set is given by

$$\widetilde{\mathrm{L}}^{+}(x) = \{y \in \Omega : \widetilde{\pi}(x, t_n) \overset{n\to+\infty}{\longrightarrow} y, \text{ for some } t_n \to +\infty\},$$

that is, the elements of $\widetilde{\mathrm{L}}^{+}(x)$ are taken in Ω and not in X.

In the present paper, we extend the LaSalle's results from [4] in the following manner. We consider impulsive semidynamical systems of type $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ subject to impulse action which varies in time, where X is a metric space, (X, π) is a semidynamical system, M is a non-empty closed subset of X that denotes the impulsive set and $\mathrm{I} : M \to \mathrm{X}$ is the impulse function. We consider the closure of $\widetilde{\pi}^{+}(x)$ in X and the elements of $\widetilde{\mathrm{L}}^{+}(x)$ in X. Due to this apparent slight difference (i.e when we consider the closure of a set in X rather than in Ω) a new phenomenon that is not present in the impulsive systems considered by [4] can occur in our impulsive systems. For instance, Example 3.1 from [7], shows that the limit set is not necessarily positively invariant under $\widetilde{\pi}$.

In the first part of the paper, we present the basis of the theory of impulsive semidynamical systems. In Section 2.1, we give some basic definitions and notations about impulsive semidynamical systems. In Section 2.2, we discuss the continuity of a function which describes the times of reaching the impulsive set. In Section 2.3, we give some additional useful definitions.

The second part of the paper concerns the main results. We introduce the concept of a Lyapunov function defined in X and we present a version of the Theorems of the Invariance and Asymptotic stability for our impulsive system.

2. Impulsive semidynamical systems

In this section, we present the basic definitions and notation of the theory of impulsive semidynamical systems.

2.1. Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(\mathrm{X}, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : \mathrm{X} \times \mathbb{R}_+ \longrightarrow \mathrm{X}$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in \mathrm{X}$ and $t, s \in \mathbb{R}_+$. We denote such system by $(\mathrm{X}, \pi, \mathbb{R}_+)$ or simply (X, π). For every $x \in \mathrm{X}$, we consider the continuous function $\pi_x : \mathbb{R}_+ \longrightarrow \mathrm{X}$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* passing through the point x, and the set $\sum_x^+ = \pi(x, \mathbb{R}_+)$ is called the positive semi-trajectory of this motion.

Let (X, π) be a semidynamical system. Given $x \in \mathrm{X}$, the *positive orbit* of x is given by $\pi^{+}(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. For $t \geq 0$ and $x \in \mathrm{X}$, we define $\mathrm{F}(x, t) = \{y : \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset \mathrm{X}$, we define $\mathrm{F}(D, \Delta) = \bigcup\{\mathrm{F}(x, t) : x \in D \text{ and } t \in \Delta\}$. A point $x \in \mathrm{X}$ is called an *initial point*, if $\mathrm{F}(x, t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X such that for every $x \in \mathrm{M}$, there exists $\varepsilon_x > 0$ such that

$$\mathrm{F}(x, (0, \varepsilon_x)) \cap \mathrm{M} = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap \mathrm{M} = \emptyset,$$

and a continuous function $\mathrm{I} : \mathrm{M} \to \mathrm{X}$ whose action we explain below in the description of the impulsive semi-trajectory of an impulsive semidynamical system. The points of M are isolated in every trajectory of the system (X, π). The set M is called the *impulsive set*, the function I is called *impulse function* and we write $\mathrm{N} = \mathrm{I}(\mathrm{M})$. We also define $\mathrm{M}^{+}(x) = (\pi^{+}(x) \cap \mathrm{M}) \setminus \{x\}$.

Given an impulsive semidynamical systems $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ and $x \in \mathrm{X}$ with $\mathrm{M}^{+}(x) \neq \emptyset$, it is always possible to find a smallest number s such that the semi-trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. For every $x \in X$, there is a positive number s, $0 < s \leq +\infty$, such that $\pi(x, t) \notin \mathrm{M}$, whenever $0 < t < s$, and $\pi(x, s) \in \mathrm{M}$ if $\mathrm{M}^{+}(x) \neq \emptyset$.

Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \mathrm{X}$. Define a function $\phi : \mathrm{X} \to (0, +\infty]$ in the following manner

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in \mathrm{M} \text{ and } \pi(x, t) \notin \mathrm{M} \text{ for } 0 < t < s, \\ +\infty, & \text{if } \mathrm{M}^{+}(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the semi-trajectory of x meets M. Thus for each $x \in \mathrm{X}$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive semi-trajectory* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is an X-valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such semi-trajectory follows inductively, as described in the following lines.

If $\mathrm{M}^{+}(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x, t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However, if $\mathrm{M}^{+}(x) \neq \emptyset$, then there is a smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in \mathrm{M}$ and $\pi(x, t) \notin \mathrm{M}$, for $0 < t < s_0$. Then we define $\widetilde{\pi}_x$ on $[0, s_0]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0 \\ x_1^{+}, & t = s_0, \end{cases}$$

where $x_1^{+} = \mathrm{I}(x_1)$ and $\phi(x) = s_0$.