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Locally Lipschitz selections in Banach lattices

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A B S T R A C T

Let X be a Banach space while $(Y, \preceq)$ is a Banach lattice. We consider the class of "upper separated" set-valued functions $F : X \to 2^Y$ and investigate the problem of the existence of convex and locally Lipschitz selections of F. We discuss some applications of the selection results obtained in the paper to the theory of deterministic and stochastic differential inclusions.

1. Introduction

In general, for deterministic or stochastic differential inclusions, an appropriate kind of regularity of their multivalued structure is required. In particular, the properties such as the Lipschitz continuity, lower, upper semicontinuity or monotonicity of set-valued mappings have most often been considered (see e.g. [1,2] and references therein). One of the main reasons is that such regularities imposed on set-valued operators allow us to use results on the existence of exact or at least approximate selections having an appropriate kind of regularity, and therefore to reduce the multivalued problems to single-valued ones. Hence regular selections have attracted considerable interest as a useful tool for proving the existence of solutions of set-valued problems. In the paper, we deal with a new class of multifunctions with values in Banach lattices, which need not satisfy any of these properties mentioned above. For this class, we study necessary and sufficient conditions for the existence of convex and locally Lipschitz selections. In particular, the convexity of selections allows us to prove an infinite-dimensional version of the convex-type Sandwich Theorem. This result extends the one dimensional, necessary and sufficient conditions for the existence of a convex function which separates two given functions. On the other hand, having a local Lipschitz selection, we prove new existence results for deterministic and stochastic differential inclusions.

We begin our considerations with auxiliary definitions and facts needed in the sequel.

2. Upper separated set-valued functions in Banach lattices

Let X, Y be Banach spaces. Let K^+ denote a cone of positive elements in Y. We will use the notation $x \preceq y$ if $y - x \in K^+$. We will always assume that $(Y, \preceq)$ is an order complete Banach lattice (i.e., every nonempty and majorized subset of

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Y has a supremum in Y). A set A in a Banach lattice is called order bounded if it is contained in some order interval $[a, b] = \{y \in Y : a \preceq y \preceq b\}$. A set A in a Banach lattice is called full (or order convex) if for each $x, y \in A$ the order interval $[x, y] \subset A$.

We adjoin to Y the greatest element $+\infty$ together with the lowest element $-\infty$ and extend the vector space operations in a natural way. Let $\bar{Y} = Y \cup \{\pm\infty\}$.

Let us consider an extended function $f : X \to \bar{Y}$. Let $\mathrm{Dom} f = \{x \in X : f(x) \neq \pm\infty\}$ and define the epigraph of f by the formula

$$Epi(f) = \{(x, a) \in X \times Y : f(x) \preceq a\}.$$

Definition 1. A function $f : X \to \bar{Y}$ is order convex if for every x, y in X and $\lambda \in [0, 1]$ $f(\lambda x + (1-\lambda)y) \preceq \lambda f(x) + (1-\lambda) f(y)$.

A function $f : X \to \bar{Y}$ is locally order-Lipschitz if and only if for every $x_0 \in X$ there exists an open neighborhood U_{x_0} and a $y \in K^+$ such that $|f(x) - f(z)| \preceq y\|x - z\|$ for every $x, z \in U_{x_0}$.

Definition 2. Let $F : X \to 2^Y$ be a set-valued function. F is majorized in a neighborhood of x_0 if there exists an open neighborhood U_{x_0} and $y \in Y$ such that for each $x \in U_{x_0}$ and every $a \in F(x)$ the inequality $a \preceq y$ holds.

A function $f : X \to Y$ is called a selection of F if $f(x) \in F(x)$ for every $x \in X$.

Let $V, W : X \to \bar{Y}$ be defined by formulas

$$\begin{aligned} V(x) &= \sup\{a : a \in F(x)\} \\ W(x) &= \inf\{b : b \in F(x)\}. \end{aligned} \tag{1}$$

Let $\Pi_{F(x)}(a)$ denote the metric projection of a point $a \in Y$ onto the set $F(x)$. We define

$$\begin{aligned} \bar{V}(x) &:= \begin{cases} \Pi_{F(x)}(V(x)) & \text{for } x \in \mathrm{Dom}\, V \\ +\infty & \text{for } x \notin \mathrm{Dom}\, V \end{cases} \\ \bar{W}(x) &:= \begin{cases} \Pi_{F(x)}(W(x)) & \text{for } x \in \mathrm{Dom}\, W \\ -\infty & \text{for } x \notin \mathrm{Dom}\, W. \end{cases} \end{aligned} \tag{2}$$

Let $\mathrm{ClConv}Y$ denote the family of all closed, convex and nonempty subsets of Y. By $\mathcal{L}(X, Y)$ we denote the space of all linear and norm-continuous operators from X to Y. Below we introduce the class of set-valued functions called "upper separated" and study their connections with the problem of the existence of order convex selections.

Definition 3. A set-valued function $F : X \to \mathrm{ClConv}Y$ is upper separated if each point $(x, \bar{W}(x) - \epsilon)$ can be separated from the set $Epi\bar{V}$ in the following sense: for every $x \in X$ and each $\epsilon \in K^+ \setminus \{0\}$ there exist $A \in \mathcal{L}(X, Y)$, $a \in R^1$ and $\delta \in K^+ \setminus \{0\}$ such that for every $y \in \mathrm{Dom}\,\bar{V}$ and each $b \in K^+$ the condition

$$A(x) - A(y) + a(\bar{W}(x) - \bar{V}(y) - \epsilon - b) - \delta \in K^+ \tag{3}$$

holds.

Example 1. When $(Y, \preceq) = (R^1, \leq)$, then $\bar{W}(x) = W(x)$, $\bar{V}(x) = V(x)$ and therefore, a set-valued function F from a Banach space X into subsets of R^1 is upper separated if for every $x \in X$ and $\epsilon > 0$ there exists a hyperplane $H_{x,\epsilon}$ strongly separating a point $(x, W(x) - \epsilon)$ from the set $EpiV$ (see [3]).

Theorem 1. *Let $F : X \to \mathrm{ClConv}Y$ be majorized in a neighborhood of some point x_0. Assume that F admits an order convex selection f. If the following conditions hold:*

(i) $V(x) \in F(x)$ *for every* $x \in \mathrm{Dom}\, V$,
(ii) $W(x) \in F(x)$ *for every* $x \in \mathrm{Dom}\, W$,

then F is upper separated.

Proof. Since every Banach lattice is normal [4, Ch.V], f is order convex and majorized in the neighborhood of some point x_0, we deduce by Theorem 3.1 of [5] that f is continuous in X. By Lemma 3.2 of [5] its order subdifferential

$$\partial f(x) = \{A \in \mathcal{L}(X, Y) : A(y) - A(x) \preceq f(y) - f(x)\ \forall y \in X\} \tag{4}$$

is nonempty. Therefore, for every $x \in X$ there exists an operator $A_x \in \mathcal{L}(X, Y)$ such that

$$A_x(y) - A_x(x) \preceq f(y) - f(x) \quad \forall y \in X. \tag{5}$$

But f is a selection of F so we have $f(y) \preceq V(y)$ and $f(x) \succeq W(x)$. Therefore, for every $b \in K^+$ we get

$$A_x(y) - A_x(x) \preceq V(y) - W(x) + b, \tag{6}$$