Nonlinear Analysis 70 (2009) 3773–3780

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the Riemann problem for 2D compressible Euler equations in three pieces

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ARTICLE INFO

Article history:
Received 7 September 2007
Accepted 29 July 2008

MSC:
35L65
35L67
76N15

Keywords:
Two-dimensional Riemann problem
Compressible Euler equations
Compatibility conditions

ABSTRACT

The Riemann problem for two-dimensional isentropic Euler equations is considered. The initial data are three constants in three fan domains forming different angles. Under the assumption that only a rarefaction wave, shock wave or contact discontinuity connects two neighboring constant initial states, it is proved that the cases involving three shock or rarefaction waves are impossible. For the cases involving one rarefaction (shock) wave and two shock (rarefaction) waves, only the combinations when the three elementary waves have the same sign are possible (impossible).

1. Introduction

The Riemann problem, a kind of Cauchy problem with the simplest discontinuous initial data, is the most fundamental problem in the field of nonlinear hyperbolic conservation laws [9]. Compared to the Cauchy problem, it is much easier to study, but still reveals the basic properties of the Cauchy problem. Due to the explicit structure of the Riemann solutions, it also serves as a touchstone for numerical schemes.

The general Riemann problem for a two-dimensional system of conservation laws

$$U_t + (F(U))_x + (G(U))_y = 0, \quad U \in R^m, \tag{1.1}$$

is the following special Cauchy problem

$$U(0, x, y) = U_0(\theta), \tag{1.2}$$

where U_0 is a given vector function of one variable θ which is the polar angle in the plane.

Many pieces of work have been contributed on four-constant Riemann problems, i.e., where the initial data are four constant states in each quadrant, which are sufficient to approximate general initial data using rectangular grids. With such initial data, the Riemann problem for scalar conservation law was studied in [10,7,11]. For the Euler equations, the most important model in gas dynamics, the Riemann problem was investigated in [12,8,5,13]. Based on characteristic analysis, patterns of oblique shock reflection and numerical experiments, a set of conjectures on the structure of solutions is presented. Unfortunately, none of them has been proved due to the complicated structure of solutions. This motivates our interest in considering much simpler initial data.

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doi:10.1016/j.na.2008.07.033

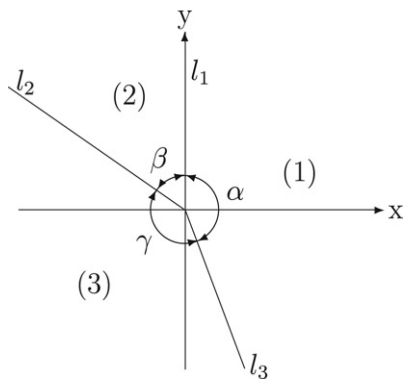

Fig. 1.

When the initial data are sectorial constants, Li and Zheng [4,6,14] proved the existence of global continuous solutions of the expansion of a wedge of gas into a vacuum for compressible Euler equations. In this paper, we are concerned with the Riemann problem in three pieces for the isentropic Euler equations:

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{1.3}$$

where ρ is the density, (u, v) is the velocity and p is the pressure given by $p = A\rho^\gamma$ where $A > 0$ is the entropy and $\gamma > 1$ is the gas constant. The initial data take three constants in three fan domains as follows:

$$(\rho, u, v)(0, x, y) = (\rho_i, u_i, v_i) \equiv (i), \quad i = 1, 2, 3. \tag{1.4}$$

For simplicity of presentation, we locate l_1 coinciding with the y-axis as in Fig. 1. This study of the two-dimensional Riemann problem is restricted to situations where exactly one wave of shock, rarefaction or contact discontinuity appears at each interface of the initial data. Then we find that the cases $3S$ and $3R$ are incompatible. For the case $R + 2S$ ($S + 2R$), only the combinations when the three elementary waves have the same sign are possible (impossible). The adiabatic Euler system with the similar initial data was considered in [3].

The outline of the paper is as follows. A preliminary is given in Section 2. Then we classify the Riemann problem according to the combinations of the elementary waves in Section 3. There we analyze which combinations are possible. Moreover, for the possible cases we give the relations that have to be satisfied by the initial data.

2. Preliminaries

Before starting with the classification of the two-dimensional Riemann problem, we briefly review the formulae for the one-dimensional elementary waves between two states.

Since both (1.3) and (1.4) are invariant under the self-similar transformation $t \to \alpha\bar{t}, x \to \alpha\bar{x}, y \to \alpha\bar{y}$ ($\alpha > 0$), we should seek the solution in the (ξ, η) plane, where $\xi = x/t, \eta = y/t$. Then the characteristic for (1.3) in the (ξ, η) plane is either

$$\lambda = \lambda_0 = \frac{\bar{v}}{\bar{u}} \quad \text{(pseudoflow characteristic)},$$

or

$$\lambda = \lambda_\pm = \frac{\bar{u}\bar{v} \pm \sqrt{c^2(\bar{u}^2 + \bar{v}^2 - c^2)}}{\bar{u}^2 - c^2} = \frac{\bar{v}^2 - c^2}{\bar{u}\bar{v} \mp \sqrt{c^2(\bar{u}^2 + \bar{v}^2 - c^2)}} \quad \text{(pseudowave characteristic)},$$

which are real and distinct if and only if $\bar{u}^2 + \bar{v}^2 > c^2$. Here $\bar{u} = u - \xi, \bar{v} = v - \eta$ and $c = \sqrt{p'(\rho)}$.

Obviously, bounded solutions must be supersonic in the neighborhood of infinity. Then, far away from the origin, the solution consists of planar elementary waves $(\rho, u, v)(\mu\xi + \nu\eta)$, which involve [2]:

(i) constant states: $(\rho, u, v) = \text{constant}$;

(ii) rarefaction waves (abbreviation R),