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Global solutions of a dynamical equation in ferrimagnet

Boling Guo [a], Xueke Pu [b,*]

[a] Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing, 100088, China

[b] The Graduate School of China Academy of Engineering Physics, P.O. Box 2101, Beijing, 100088, China

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ABSTRACT

We consider global solutions of a dynamical equation in ferrimagnet. We show that it admits a global weak solution by using the penalty method. By the energy estimates method we show there exists a unique global smooth solution. Finally we establish the relationship between this equation and wave maps.

1. Introduction

In this paper we consider the following dynamical equation in ferrimagnetic materials of the following form

$$\begin{cases} u_t = u \times \Box u + \lambda u \times (u \times \Box u), & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3) \in \mathbb{S}^2$ (i.e. the unit sphere in $\mathbb{R}^3$) is the unknown vector that maps a regular domain $\Omega \subset \mathbb{R}^d$ to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, $\times$ denotes the cross product in $\mathbb{R}^3$ and $\lambda > 0$ is a positive damping parameter.

This equation was first derived by Borisov, Kiseliev and Talutz in [2] when studying the ferrimagnetic phenomenon in ferrimagnetic materials. Just as the Landau–Lifshitz equation (see for [3–5])

$$u_t = u \times \Delta u - \lambda u \times (u \times \Delta u) \tag{1.2}$$

is important in the study of continuum ferromagnets, this equation plays a key role in the study of ferrimagnetic materials. It aroused the interests of both physicists and mathematicians greatly. Thus we should study it mathematically rigorously. Formally, this equation is very similar to the Landau–Lifshitz equation with the Laplacian operator Δ replaced by the wave operator $\Box$.

Wave maps are maps from Minkowski space $\mathscr{M}$ into a Riemannian manifold $\mathscr{N} \hookrightarrow \mathbb{R}^k$ that satisfy the wave equation with partial derivatives replaced by covariant derivatives. They are the prototypes of geometric wave equations and are studied by many authors and the interested readers can refer to [8] for more details. It can be also regarded as a wave equation from $\mathscr{M}$ to $\mathbb{R}^k$, with the range restricted to the manifold $\mathscr{N}$. When the target manifold is the unit sphere $\mathbb{S}^k \hookrightarrow \mathbb{R}^{k+1}$, the wave map reads

$$\Box u = (|\nabla u|^2 - |u_t|^2)u. \tag{WM}$$

* Corresponding author.
E-mail addresses: gbl@iapcm.ac.cn (B. Guo), xuekepu@tom.com (X. Pu).

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We should add here that this is the famous nonlinear σ-model, and it is important in the study of the dynamics of anti-ferromagnetic materials, see [6,7,9] for more details.

For the Landau–Lifshitz equation (1.2), Alouges and Soyeur [1] established the existence of weak solutions and considered the limit as the damping parameter goes to zero or to infinity. On the other hand, Guo and Hong [4] established the relationship between the Landau–Lifshitz equation and the heat flow of harmonic maps. Then two natural questions arose. One is whether there exist suitable solutions for Eq. (1.1), another one is the relationship between this equation and the wave map (WM) from Ω to $\mathbb{S}^2$. These questions are our main targets in this paper.

We will consider two important cases. The first one is when the problem is posed periodically. In this case, we denote $M = \Omega = [-L_1, L_1] \times \cdots \times [-L_d, L_d]$, and the problem (1.1) is completed with the periodical condition $u(x - L, t) = u(x + L, t)$, where the vector $L = (L_1, \ldots, L_d)$. We will consider the existence of weak solutions of the periodical problem. For this purpose, we need to make clear what a weak solution is. It is not difficult to verify that when u is regular, by taking cross product with λu, Eq. (1.1) is equivalent to

$$\frac{1}{1+\lambda^2}\partial_t u - \frac{\lambda}{1+\lambda^2} u \times \partial_t u - u \times \partial_{tt} u + u \times \Delta u = 0. \tag{1.3}$$

This leads us to introduce the following notion of a weak solution for Eq. (1.1).

Definition 1.1. Let $u_0 \in H^1(M)$, $|u_0| = 1$ a.e., and $u_1 \in L^2(M)$. We say that $u(x, t)$ is a weak solution of the problem (1.1) provided:

(i) for all $T > 0$, $u \in L^2(0, T; H^1(M))$, $u_t \in L^2(0, T; L^2(M))$, with $|u| = 1$ a.e.;
(ii) for all $\Phi \in H^1(M \times [0, T])$, with $\Phi(x, T) = 0$, then there holds:

$$\frac{1}{1+\lambda^2}\int_{M_T} u_t \cdot \Phi \mathrm{d}x\mathrm{d}t + \frac{\lambda}{1+\lambda^2}\int_{M_T} (u \times u_t) \cdot \Phi \mathrm{d}x\mathrm{d}t + \int_M (u \times u_t) \cdot \Phi|_{t=0} \mathrm{d}x$$
$$+ \int_{M_T} u \times u_t \cdot \Phi_t \mathrm{d}x\mathrm{d}t - \sum_{i=1}^m \int_{M_T} u \times \frac{\partial u}{\partial x_i} \cdot \frac{\partial \Phi}{\partial x_i} = 0; \tag{1.4}$$

(iii) $u(x, 0) = u_0(x)$ in the trace sense.

In this case, we have the following:

Theorem 1.1. *Let $u_0 \in H^1(M, \mathbb{S}^2)$ and $u_1 \in L^2(M, T_u\mathbb{S}^2)$. Then there exists a global weak solution of the problem (1.1) with initial data (u_0, u_1).*

The second case is when the problem is posed in $\mathbb{R}^d$. For this Cauchy problem, we can only have the local well-posedness result in dimension $d \geq 2$, however when we restrict ourselves to dimension $d = 1$, we have the global well-posedness result.

Theorem 1.2. *For any data $(u_0, u_1) \in L^2_{\mathrm{loc}}(\mathbb{R}) \times H^1(\mathbb{R})$, such that ∇u_0 belongs to $H^1(\mathbb{R})$, $|u_0| = 1$ a.e., there exists a unique global smooth solution u of class H^2 which preserves the regularity of the initial data.*

We remark that the existence of global smooth solutions in the periodical case can be established as the same and we will not handle it explicitly in this paper.

In the final section, we show that when the damping parameter $\lambda \to \infty$, the solution approximates to the wave map that takes values in $\mathbb{S}^2$. Thus we find new interpretations of the wave maps, and this may link the mathematical studies of wave maps to applications in physics.

Throughout this paper, $Du = (u_t, \nabla u)$ always denotes the space–time derivative of u, and $T_u\mathbb{S}^2$ denotes the tangent space of $\mathbb{S}^2$ at u. We use $\|\cdot\|$ to denote the L^2-norm and $\|\cdot\|_X$ to denote the X-norm on either M or $\mathbb{R}^d$. M_T always denotes the product $M \times [0, T]$ and C denotes a constant which can vary from line to line.

This paper is organized as follows. In the next section, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2. Finally in Section 4, we show its relationship with wave maps.

2. Proof of Theorem 1.1

In this section, we focus ourselves on the existence of global weak solutions of Eq. (1.1). For this purpose, we deduce two equivalent forms of this equation. One is (1.3) and the other one is by taking the cross product of Eq. (1.3) with u,

$$\Box u + \frac{\lambda}{1+\lambda^2}\partial_t u + \frac{1}{1+\lambda^2} u \times \partial_t u + (|\partial_t u|^2 + |\nabla u|^2)u. \tag{2.1}$$

We then construct the following penalized problem of this equation

$$\partial_{tt} u^k - \Delta u^k + \frac{\lambda}{1+\lambda^2}\partial_t u^k + \frac{1}{1+\lambda^2} u^k \times \partial_t u^k + k(|u^k|^2 - 1)u^k = 0, \tag{2.1$_k$}$$

completed with initial data $u^k(x, 0) = u_0$ and $u^k_t(x, 0) = u_1$.