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Attractor and spatial chaos for the Brusselator in R^N

Boling Guo, Yongqian Han *

Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, PR China

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ABSTRACT

Brusselator is an important dynamical system which can be described by a reaction–diffusion system. In this paper it is proved that this reaction–diffusion system possesses a global attractor $\mathcal{A}$ in the corresponding phase space. The upper and lower bounds of Kolmogorov ε-entropy of $\mathcal{A}$ are obtained.

Moreover we give a more detailed study of spatial chaos of the attractor $\mathcal{A}$ for the Brusselator in R^N. We interpret a group of spatial shifts as a dynamical system which acts on the attractor $\mathcal{A}$. By using the technique of unstable manifolds, it is proved that this dynamical system is chaotic. In order to clarify the nature of this chaos, we construct the Lipschitz-continuous homeomorphic embedding of a typical model dynamical system whose chaotic behavior is evident, into the spatial shifts on the attractor $\mathcal{A}$. This typical dynamical system generalizes the symbolic system. It was first introduced by Zelik.

1. Introduction

We consider the following Cauchy problem of a reaction–diffusion system

$$u_t = d\Delta u - (b+1)u + u^2 v + a, \tag{1.1}$$

$$v_t = d\Delta v + bu - u^2 v, \quad x \in R^N,\ t > 0, \tag{1.2}$$

$$u(x, t=0) = u_0(x) \geq 0, \qquad v(x, t=0) = v_0(x) \geq 0. \tag{1.3}$$

This system is a model of a certain chemical morphogenetic process due to Turing [23]. It is named Brusselator. Here d, a and b are strictly positive constants, $N = 1, 2, 3$.

System (1.1) and (1.2) has been extensively studied, see [1,3,9,11,13,16,17,19,20,24, etc.], and references therein. In particular, system (1.1) and (1.2) was applied to study oscillatory Turing pattern [23], stationary pattern selection and competition [3]. Hollis, Martin and Pierre [11] proved existence of a global bounded solution of (1.1) and (1.2) with initial boundary conditions in the bounded domain of R^N. For fixed v_0, the global attractor of u for problem (1.1)–(1.3) was established in [9].

To study the spatial chaos of problem (1.1)–(1.3), the classical Sobolev spaces (such as $H^l(R^N)$ $(l \geq 1)$) do not seem to be adequate because nondecaying initial data and a number of natural structures from the physical point of view (such as spatially periodic solutions, travelling waves, static states, etc.) are out of consideration. For example, it is obvious that the static state $(u_s, v_s) = (a, b/a)$ is a solution of (1.1) and (1.2), but $u_s, v_s \notin C([0,T]; H^l(R^N))$. Therefore we discuss system (1.1)–(1.3) in some weighted Sobolev spaces such as [7,26,27, etc.].

* Corresponding author.
E-mail address: han_yongqian@iapcm.ac.cn (Y. Han).

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Definition 1.1 ([27]). A function $\phi \in C_{loc}(R^N)$ is a weight function with the growth rate $\alpha \geq 0$ if the conditions $\phi(x+y) \leq C_\phi e^{\alpha|x|}\phi(y)$ and $\phi(x) > 0$ are satisfied for every $x, y \in R^N$.

Definition 1.2 ([27]). Let ϕ be a weight function with the growth rate α. We define the spaces

$$W_\phi^{l,p}(R^N) = \left\{ u \in \mathcal{D}'(R^N) : \|u\|^p_{\phi,l,p} \equiv \sum_{|\beta|\leq l} \int_{R^N} \phi(x)|\partial_x^\beta u(x)|^p dx < \infty,\ l \in \mathcal{N} \right\},$$

$$W_{b,\phi}^{l,p}(R^N) = \left\{ u \in \mathcal{D}'(R^N) : \|u\|^p_{b,\phi,l,p} \equiv \sup_{x_0\in R^N} \phi(x_0)\|u, B^1_{x_0}\|^p_{l,p} < \infty,\ l \in \mathcal{N} \right\}.$$

We will write $W_b^{l,p}$ instead of $W_{b,1}^{l,p}$.

It is obvious that the static state $(u_s, v_s) = (a, b/a)$ of (1.1) and (1.2) belongs to $W_b^{l,p}(R^N)$.

The semi-linear system of parabolic equations

$$u_t = a\Delta_x u + \lambda_0 u - f(u, \nabla u) + g(x), \quad x \in \Omega, \tag{1.4}$$

$$u|_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0, \tag{1.5}$$

in an unbounded domain Ω has been studied in [4,7,26,27] and references therein. If the nonlinear function $f(u, \nabla u)$ satisfies the dissipation assumption

$$f(u, \nabla u) \cdot u \geq -C, \tag{1.6}$$

the global attractor of (1.4) and (1.5) has been constructed, the upper and lower bounds of Kolmogorov ε-entropy of the infinite-dimensional global attractor for Eqs. (1.4) and (1.5) have been obtained, and the spatial complexity and spatial chaos of the global attractor for Eqs. (1.4) and (1.5) have been studied in [7,26,27] and references therein. Unfortunately, the nonlinearity of Eqs. (1.1) and (1.2) does not satisfy the dissipation assumption (1.6).

The well-posedness of some reaction–diffusion systems in R^N has been extensively studied by many authors, see [6,10,18, etc.] and references therein.

For system (1.1)–(1.3), the global well-posedness of the solution in $L^\infty\left([0,\infty); W_b^{2,2}(R^N)\right)$ has been established in [9]. In this paper we construct the attractor $\mathcal{A}$ of system (1.1)–(1.3) which is bounded in $W_b^{2,2}(R^N)$ and compact in a local topology of $W_{loc}^{2,2}(R^N)$. Although Hausdorff dimension and fractal dimension of this attractor $\mathcal{A}$ may be infinite, the upper and lower bounds of Kolmogorov ε-entropy of the attractor $\mathcal{A}$ are obtained. In order to study the spatial chaos of the attractor $\mathcal{A}$, we interpret a group of spatial shifts as a dynamical system which acts on the attractor $\mathcal{A}$. By using the technique of unstable manifolds, it is proved that this dynamical system is chaotic. In order to clarify the nature of this chaos, we construct the Lipschitz-continuous homeomorphic embedding of a typical model dynamical system whose chaotic behavior is evident, into the spatial shifts on the attractor $\mathcal{A}$. This typical dynamical system generalizes the symbolic system. It was first suggested by Zelik [7,26,27].

Throughout this paper, in order to simplify the exposition, different positive constants might be denoted by the same letter C; if necessary, by $C(\cdot,\cdot)$ we denote the constant depending only on the quantities appearing in parentheses.

The plan of this paper is as follows. In Section 2, we derive several a priori estimates for the solutions of (1.1)–(1.3), and establish the global well-posedness of the solution again. In Section 3, we construct the global attractor $\mathcal{A}$, and obtain the upper bound of Kolmogorov ε-entropy of attractor $\mathcal{A}$. In Section 4, we study the spatial chaos of attractor $\mathcal{A}$, and obtain the lower bound of Kolmogorov ε-entropy of attractor $\mathcal{A}$.

2. A priori estimates, existence and uniqueness of solution

In this section, our main task is to improve a priori estimates in [9]. First we recall the properties of several classes of Sobolev spaces in unbounded domains, which will be used blow. For a detailed study of these spaces, see [7,26,27].

Proposition 2.1 ([27]). *Let ϕ be a weight function with the growth rate α, and R be a positive number. Then the following estimates are valid:*

$$C_2 \int_{R^N} \phi(x)|u(x)|^p dx \leq \int_{R^N} \phi(x_0) \int_{B^R_{x_0}} |u(x)|^p dx\, dx_0 \leq C_1 \int_{R^N} \phi(x)|u(x)|^p dx. \tag{2.1}$$

Corollary 2.2 ([27]). *The equivalent norm in a weighted Sobolev space $W_\phi^{l,p}(R^N)$ is given by*

$$\|u\|_{\phi,l,p} = \left(\int_{R^N} \phi(x_0)\|u, B^R_{x_0}\|^p_{l,p} dx_0 \right)^{1/p}. \tag{2.2}$$

In particular, norms (2.2) are equivalent for different $R \in R_+$.