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On Joachimsthal's theorems in semi-Euclidean spaces

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ABSTRACT

In this work, we study Joachimsthal's theorems in semi-Euclidean spaces E_ν^{n+1}. We obtain a relation between the curvatures of the strips in semi-Euclidean spaces E_ν^{n+1} in matrix form depending on a semi-orthogonal matrix. This matrix equation gives us a generalization of the original Joachimsthal theorem in semi-Euclidean space E_1^3 and the well-known Joachimsthal theorem of the surface theory. In addition, we reobtain the well-known Joachimsthal theorem from our matrix equation in the case $n = 2$, $\nu = 0$ and the original Joachimsthal theorem from our matrix equation in the case $n = 2$, $\nu = 1$.

1. Introduction

Because of relativistic theories, indefinite metrics are beginning to have significance in theoretical physics, and so this framework is very important in differential geometry. Besides Lorentz metrics and low dimensions, other semi-Riemannian metrics and higher dimensions are applicable in new theories in physics, for example, $(3+2)$-dimensional relativity, string and superstring theories.

As is known, curves and surfaces are some of the fundamental objects of differential geometry. The notion of a strip enables us to study the behaviors of these two geometric objects together. Moreover, Joachimsthal's theorems give us an interpretation of two surfaces which intersect.

The higher curvature of a curve in E^m is studied by Gluck [6] and Sabuncuoğlu and Hacısalihoğlu [9]. The higher curvature of a strip in E^m is calculated by Sabuncuoğlu and Hacısalihoğlu [10]. Joachimsthal's theorems in Euclidean spaces E^m are given by Sabuncuoğlu [11], but to our knowledge, there has been no study on Joachimsthal's theorems in semi-Euclidean spaces E_ν^{n+1}. Such a study is the object of this paper. We consider nondegenerate hypersurfaces of a semi-Euclidean space, and in the lightlike case there occur some difficulties (see [5], for instance) such as the absence of the notion of an angle, and this case is still open.

In Section 2 we shall give basic concepts for Lorentz geometry, semi-Riemannian manifolds, isometries, semi-Riemannian submanifolds, semi-orthogonal groups, and structure equations.

In Section 3 we shall calculate the higher curvature of a curve in the semi-Euclidean spaces E_ν^{n+1}; then we derive the relations between the higher curvature of a strip and the higher curvature of the curve of the strip in the semi-Euclidean spaces.

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In Section 4 we derive a relation between the curvatures of the strips $(\alpha(S), M)$ and $(\alpha(S), M')$ in the matrix form

$$[\bar{t}'_{ij}] = \varepsilon H^{-1}\varepsilon\,[\bar{t}_{ij}]\varepsilon H\varepsilon + \frac{\psi_1^*(\mathrm{d}H^t)}{\mathrm{d}s}\varepsilon H\varepsilon$$

where H is the semi-orthogonal matrix which corresponds to the linear mapping from the base fields system $\{X_1, \bar{X}_2, \ldots, \bar{X}_\nu, \bar{X}_{\nu+1}, \ldots, \bar{X}_{n+1}\}$ and $\{X_1, \bar{X}'_2, \ldots, \bar{X}'_\nu, \bar{X}'_{\nu+1}, \ldots, \bar{X}'_{n+1}\}$. We show that this matrix equation gives us a generalization of the original Joachimsthal theorem in the semi-Euclidean space E_1^3 and the well-known Joachimsthal theorem of surface theory.

Finally, in Section 5, we shall obtain the following special cases of the Joachimsthal theorem in semi-Euclidean spaces:

(1) $n = m - 1, \quad \nu = 0,$

(2) $n = 2, \quad \nu = 1,$

(3) $n = 2, \quad \nu = 0.$

2. Preliminaries

As is known, a real vector space V with a symmetric bilinear form $\langle , \rangle : V \times V \to V$ with the property, called nondegeneracy,

$$\langle v, w\rangle = 0 \quad \text{for all } w \in V \text{ implies } v = 0$$

is called a scalar product space. Let $v \in V$; then v is called a *spacelike, timelike and lightlike (or null)* vector if even $v = 0$, $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$. These are called the *causal characters* of a vector and as is known there occur many difficulties arising from the existence of these characters. $\mathbb{R}^n$ has a scalar product defined by

$$\langle v, w\rangle = \sum_{i=1}^{n-\nu+1} v^i w^i - \sum_{i=n-\nu+2}^{n+1} v^i w^i$$

where v^i, w^i, $1 \le i \le n$, are the natural coordinates of v and w.

A manifold with a smooth choice, the *metric tensor*, of scalar products associated with each of its tangent spaces is called a *semi-Riemannian manifold.* For details on semi-Riemannian geometry we refer the reader to [8].

Let E_ν^{n+1} denote the $(n+1)$-dimensional Euclidean space E^{n+1} with metric tensor $\langle , \rangle$ defined by

$$\mathrm{d}s^2 = \sum_{i=1}^{n-\nu+1} \mathrm{d}x_i^2 - \sum_{i=n-\nu+2}^{n+1} \mathrm{d}x_i^2$$

where $(x_1, \ldots, x_{n+1})$ is a rectangular coordinate system of E^{n+1}. Then E_ν^{n+1} is called a *semi-Euclidean space* (see [12]).

Let us identify the set of all linear isometries $T_p(E_\nu^{n+1}) \longrightarrow T_p(E_\nu^{n+1})$ as the set $O(\nu, n-\nu+1)$ of all matrices $g \in GL(n+1, R)$ that preserve the scalar product $\langle , \rangle$. As is known (see [8]) $O(\nu, n-\nu+1)$ is a Lie subgroup of $GL(n+1, R)$ and $g \in O(\nu, n-\nu+1)$ if and only if ${}^t g = \varepsilon g^{-1}\varepsilon$, where

$$\varepsilon = \begin{bmatrix} I_{n-\nu+1} & O_{(n-\nu+1)\times\nu} \\ O_{\nu\times(n-\nu+1)} & -I_\nu \end{bmatrix}.$$

The following examples are due to [2,8].

Example 1. For each $\theta \in \mathbb{R}$, the orthogonal matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a rotation of $\mathbb{R}^2$ through (oriented) angle θ. The function $\theta \to R_\theta$ is a smooth homomorphism from $\mathbb{R}$, under addition, into $O(2)$. Its kernel is $2\pi\mathbb{Z}$ and its image is the rotation group $O^+(2)$, the component of the identity in $O(2)$. Thus $O^+(2)$ and its other coset $O^-(2)$ are diffeomorphic to circles.

Example 2. For each $\varphi \in \mathbb{R}$, the semi-orthogonal matrix

$$B_\varphi = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$$

is called a boost of $\mathbb{R}_1^2$ through (oriented) Lorentzian angle φ. As above, $\varphi \to B_\varphi$ is a homomorphism, but in this case it is one-to-one. Any $a \in O_1(2)$ must carry each hyperbola $\langle p, p\rangle = 1$ and $\langle p, p\rangle = -1$ into itself but may reverse the branches of each. These two choices split $O_1(2)$ into four disjoint open subsets. The one preserving all branches is exactly the set B of all boosts. B is a subgroup diffeomorphic to $\mathbb{R}^1$ and is thus the component of the identity in $O_1(2)$. The other three sets are cosets of B; hence $O_1(2)$ has four components, each diffeomorphic to $\mathbb{R}^1$.