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On a nonlinear wave equation associated with the boundary conditions involving convolution

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ABSTRACT

The paper deals with the initial boundary value problem for the linear wave equation

$$\begin{cases} u_{tt} - \dfrac{\partial}{\partial x}(\mu(x,t)u_x) + \lambda u_t + F(u) = 0, \quad 0 < x < 1, 0 < t < T, \\ \mu(0,t)u_x(0,t) = g_0(t) + \displaystyle\int_0^t k_0(t-s)u(0,s)\mathrm{d}s, \\ -\mu(1,t)u_x(1,t) = g_1(t) + \displaystyle\int_0^t k_1(t-s)u(1,s)\mathrm{d}s, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{cases} \tag{1}$$

where F, μ, g_0, g_1, k_0, k_1, u_0, u_1 are given functions and λ is a given constant. The paper consists of four main parts. In Part 1, under conditions $(u_0, u_1, g_0, g_1, k_0, k_1) \in H^1 \times L^2 \times (H^1(0,T))^2 \times (W^{1,1}(0,T))^2$, $\mu \in W^1(Q_T)$, $\mu_t \in L^1(0,T;L^\infty)$, $\mu(x,t) \geq \mu_0 > 0$, a.e. $(x,t) \in Q_T$; the function F continuous, $\int_0^z F(s)\mathrm{d}s \geq -C_1z^2 - C_1'$, for all $z \in \mathbb{R}$, with C_1, $C_1' > 0$ are given constants and some other conditions, we prove that, the problem (1) has a unique weak solution u. The proof is based on the Faedo–Galerkin method associated with the weak compact method. In Part 2 we prove that the unique solution u belongs to $H^2(Q_T) \cap L^\infty(0,T;H^2) \cap C^0(0,T;H^1) \cap C^1(0,T;L^2)$, with $u_t \in L^\infty(0,T;H^1)$, $u_{tt} \in L^\infty(0,T;L^2)$, if we assume $(u_0, u_1) \in H^2 \times H^1$, $F \in C^1(\mathbb{R})$ and some other conditions. In Part 3, with $F \in C^{N+1}(\mathbb{R})$, $N \geq 2$, we obtain an asymptotic expansion of the solution u of the problem (1) up to order $N+1$ in a small parameter λ. Finally, in Part 4, we prove that the solution u of this problem is stable with respect to the data $(\lambda, \mu, g_0, g_1, k_0, k_1)$.

1. Introduction

In this paper, we consider the initial boundary value problem for the nonlinear wave equation

$$u_{tt} - \frac{\partial}{\partial x}(\mu(x,t)u_x) + f(u,u_t) = 0, \quad 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$\mu(0,t)u_x(0,t) = g_0(t) + \int_0^t k_0(t-s)u(0,s)\mathrm{d}s, \tag{1.2}$$

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$$-\mu(1,t)u_x(1,t) = g_1(t) + \int_0^t k_1(t-s)u(1,s)\mathrm{d}s, \tag{1.3}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \tag{1.4}$$

where $f(u, u_t) = F(u) + \lambda u_t$, with λ is a constant, $F, \mu, g_0, g_1, k_0, k_1, u_0, u_1$ are given functions satisfying conditions specified later.

In [1], An and Trieu studied a special case of problem (1.1) and (1.4) associated with the following boundary conditions:

$$u_x(0,t) = g_0(t) + h_0 u(0,t) + \int_0^t k_0(t-s)u(0,s)\mathrm{d}s, \tag{1.5}$$

$$u(1,t) = 0, \tag{1.6}$$

with $\mu \equiv 1, u_0 = u_1 \equiv 0$, and $f(u, u_t) = Ku + \lambda u_t$, with $K \geq 0, \lambda \geq 0, h_0 > 0$ given constants and g_0, k_0 are given functions. In the latter case, the problem (1.1) and (1.4)–(1.6) is a mathematical model describing the shock of a rigid body and a linear viscoelastic bar resting on a rigid base [1].

In [2] Bergounioux, Long and Dinh studied problem (1.1) and (1.4) with the mixed boundary conditions (1.2) and (1.3) standing for

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)\mathrm{d}s, \tag{1.7}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.8}$$

where $f(u, u_t) = Ku + \lambda u_t$, with $K \geq 0, \lambda \geq 0, h \geq 0, K_1 \geq 0, \lambda_1 > 0$ are given constants and g, k are given functions.

In [11], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the problem (1.1), (1.4), (1.7) and (1.8) for the case of $f(u, u_t) = K|u|^{p-2}u + \lambda\, |u_t|^{q-2}\, u_t$, where $K, \lambda \geq 0, p, q \geq 2$ and $(u_0, u_1) \in H^2 \times H^1$.

In [12], Long, Ut and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1) and (1.4) with the mixed boundary conditions (1.2) and (1.3) standing for

$$u(0,t) = 0, \tag{1.9}$$

$$-\mu(t)u_x(1,t) = g(t) + K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - \int_0^t k(t-s)u(1,s)\mathrm{d}s, \tag{1.10}$$

where $\mu \equiv \mu(t), u_0 \in H^2$ and $u_1 \in H^1, f(u, u_t) = Ku + \lambda u_t$, with $\lambda_1(t) \geq \lambda_0 > 0, K_1(t), g(t), k(t)$ are given functions, and $K \geq 0, \lambda \geq 0, \lambda_0 > 0$ are given constants. In this case, the problem (1.1), (1.4), (1.9) and (1.10) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

In [14] Santos studied the asymptotic behavior of solution of problem (1.1), (1.4) and (1.9), with $f(u, u_t) \equiv 0, \mu = \mu(t)$, associated with a boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)\mathrm{d}s = 0, \quad t > 0. \tag{1.11}$$

Santos transformed (1.11) into (1.10) with $K_1(t) = \frac{g'(0)}{g(0)}, \lambda_1(t) = \frac{1}{g(0)}$ are positive constants.

In this paper, we consider four main parts. In Part 1, under conditions $(u_0,\ u_1, g_0, g_1, k_0, k_1) \in H^1 \times L^2 \times \left(H^1(0,T)\right)^2 \times \left(W^{1,1}(0,T)\right)^2$, $\mu \in W^1(Q_T)$, $\mu_t \in L^1(0,T;L^\infty)$, $\mu(x,t) \geq \mu_0 > 0$, a.e. $(x,t) \in Q_T$; the function F continuous, $\int_0^z F(s)\mathrm{d}s \geq -C_1 z^2 - C_1'$, for all $z \in \mathbb{R}$, with $C_1, C_1' > 0$ are given constants and some other conditions, we prove that the problem (1.1)–(1.4) has a unique weak solution u. The proof is based on the Faedo–Galerkin method associated with the weak compact method. We remark that the linearization method in the paper [10] cannot be used in [9,13]. In Part 2 we prove that the unique solution u belongs to $H^2(Q_T) \cap L^\infty(0,T;H^2) \cap C^0(0,T;H^1) \cap C^1(0,T;L^2)$, with $u_t \in L^\infty(0,T;H^1)$, $u_{tt} \in L^\infty(0,T;L^2)$, if we assume $(u_0, u_1) \in H^2 \times H^1, F \in C^1(\mathbb{R})$ and some other conditions. In Part 3, with $F \in C^{N+1}(\mathbb{R})$, $N \geq 2$, we obtain an asymptotic expansion of the solution u of the problem (1.1)–(1.4) up to order $N+1$ in a small parameter λ. Finally, in Part 4, we prove that the solution u of this problem is stable with respect to the data $(\lambda, \mu, g_0, g_1, k_0, k_1)$. The results obtained here may be considered as the generalizations of those in [1,2,9–14].

2. The existence and uniqueness theorem

Put $\Omega = (0,1), Q_T = \Omega \times (0,T), T > 0$. We omit the definitions of usual function spaces: $C^m\left(\overline{\Omega}\right), L^p(\Omega), W^{m,p}(\Omega)$.

We denote $W^{m,p} = W^{m,p}(\Omega), L^p = W^{0,p}(\Omega), H^m = W^{m,2}(\Omega), 1 \leq p \leq \infty, m = 0, 1, \ldots$.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar products of continuous linear functionals with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and