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Stability and asymptotic behavior of solutions for some linear partial functional differential equations in critical cases

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ABSTRACT

In this work, we study the stability of the solution semigroup for some linear partial functional differential equations with infinite delay in a Banach space when the exponential stability fails. We use the so-called characteristic equation to compute the order of each pole of the resolvent operator associated with the infinitesimal generator of the solution semigroup. This result allows us to give sufficient conditions for having stability of the solution semigroup.

1. Introduction

The aim of this work is to study the stability and the asymptotic behavior of the solution semigroup associated with the following partial functional differential equation with infinite delay:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = Au(t) + L(u_t) & \text{for } t \geq 0, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1}$$

where A is a closed linear operator on a Banach space X, and we assume that A satisfies the following Hille–Yosida conditions:

($\mathbf{H}_1$) there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N} \text{ and } \lambda > \omega\} < \infty,$$

where $\rho(A)$ is the resolvent set of A.

$\mathcal{B}$ is a normed space of functions mapping $(-\infty, 0]$ to X satisfying some fundamental axioms introduced by Hale and Kato [10]. Those axioms will be given in Section 2. For every $t \geq 0$, as usual, the history function u_t is a function mapping $(-\infty, 0]$ into X defined by

$$u_t(\theta) = u(t + \theta) \quad \text{for } \theta \in (-\infty, 0]$$

and L is a bounded linear operator from $\mathcal{B}$ to X.

The stability has been the focus of various studies designed to establish the asymptotic behavior of solutions. In the case of ordinary and partial differential equations, the stability of stationary solutions has been extensively studied; for instance, we refer the reader to the monograph [7]. In the case of functional differential equations with finite delay, the phase space is

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generally the space of continuous functions mapping $[-r, 0]$ into X provided with the uniform norm topology or the space of X-valued Bochner integrable functions on $[-r, 0]$; see [5,15] and the references therein. For functional differential equations with infinite delay, the situation is different, since the choice of the phase space $\mathcal{B}$ plays an important role in studying the qualitative behavior of solutions, and many tools fail and other theories and techniques cannot be applied; for more details we refer the reader to [1–4,8–13] and the references therein. In [2], the authors studied the existence, the regularity and the exponential stability of equilibrium; the authors gave an explicit characterization of the infinitesimal generator $\mathcal{A}_L$ of the solution semigroup, which has been used to study the exponential stability of the semigroup solution, and they proved that when $\mathcal{B}$ is a uniform fading memory space the exponential stability is completely obtained by solving the so-called characteristic equation

$$\ker \Delta(\lambda) \neq \{0\} \quad \text{for } \mathrm{Re}(\lambda) < 0,$$

where the operator $\Delta(\lambda) : D(A) \to X$ is given by

$$\Delta(\lambda) = \lambda \mathrm{I} - A - L(\mathrm{e}^{\lambda \cdot}\mathrm{I}).$$

In [13], the authors investigated the stability for when A is the infinitesimal generator of a strongly continuous semigroup. They studied the stability in critical cases, namely where the characteristic equation has solutions on the imaginary axis.

Recently in [9], we established a new representation of the infinitesimal generator $\mathcal{A}_L$ of the solution semigroup of Eq. (1) for when A is a Hille–Yosida operator. We proved that $\varphi \in D(\mathcal{A}_L)$ if and only if the following assertions are satisfied:

(i) $\varphi(0) \in D(A)$,
(ii) $A\varphi(0) + \mathrm{F}(\varphi) \in \overline{D(A)}$,
(iii) there exists (then for all) $\psi \in \mathcal{B} \cap \mathcal{C}^1((-\infty, 0]; X)$ such that $\psi' \in \mathcal{B}$, $\psi(0) \in \overline{D(A)}$, $\varphi - \psi \in D(B_S)$ and $\psi'(0) = A\varphi(0) + L(\varphi)$.

If φ satisfies (i), (ii) and (iii), then

$$\mathcal{A}_L\varphi = B_S(\varphi - \psi) + \psi'.$$

Here B_S is the infinitesimal generator of the solution semigroup of the following trivial equation:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = 0 \quad \text{for } t \geq 0, \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \tag{2}$$

Moreover in [9], we discussed the exponential stability of the solution semigroup; we showed that if all characteristic values have negative real part, then the solution semigroup is exponentially stable and, on the other hand, if there exists a characteristic value with positive real part, then the solution semigroup is unstable.

In the present work, we focus our attention on the critical case when Eq. (1) has characteristic values λ with $\mathrm{Re}(\lambda) = 0$ and all the others have negative real parts. As is well known, we cannot immediately reach a conclusion on the behavior of the solutions. In fact both stability and instability may hold. Here, we characterize eigenvalues and their associated eigenfunctions. Furthermore, we compute the resolvent operator $(\lambda - \mathcal{A}_L)^{-1}$ of the generator $\mathcal{A}_L$ in terms of $\Delta(\lambda)^{-1}$ and $(\lambda - B_S)^{-1}$. This result allows us to discuss the stability of the solution semigroup. The present work is a continuation of [2] and [9] to studying the stability in the critical case and is an extension of [13].

This work is organized as follows. In Section 2, we collect some fundamental axioms of the phase space $\mathcal{B}$; also we recall some results about the existence of solutions as well as the semigroup aspect of Eq. (1) that will be used in this work. Section 3 is reserved for the characterization of the eigenvalues of $\mathcal{A}_L$ and its eigenfunctions. In Section 4, we compute the resolvent operator $R(\lambda, \mathcal{A}_L)$ of $\mathcal{A}_L$. In Section 5, we discuss the stability of the solution semigroup of Eq. (1). The last section is devoted to studying the stability for a Lotka–Volterra model with diffusion.

2. Preliminary results

In this work, we employ an axiomatic definition of the phase space $\mathcal{B}$ which was first introduced by Hale and Kato [10]. We assume that $(\mathcal{B}, \|\cdot\|)$ is a normed space of functions mapping $(-\infty, 0]$ into X satisfying the following assumptions:

(**A**) If a function $x : (-\infty, \sigma + a] \longrightarrow \mathrm{X}$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ such that $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a]$, the following properties hold:

(i) $x_t \in \mathcal{B}$,
(ii) $|x(t)| \leq H\|x_t\|$,
(iii) $\|x_t\| \leq K(t - \sigma)\sup\{|x(s)| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|$,
(iv) $t \to x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a]$,

where H is a positive constant, $K(\cdot)$, $M(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are two functions with K continuous and M locally bounded.

(**B**) The space $\mathcal{B}$ is complete.