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Continuous embeddings and continuation methods☆

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ABSTRACT

Sufficient conditions are given to assert that a compact mapping defined on a continuous embedding between Hilbert spaces over $\mathbb{K}$ has a fixed point. The proof of the result is based upon continuation methods.

1. Preliminaries

Let X and Y be two Banach spaces. If $u : X \to Y$ is a continuous mapping, then one way of solving the equation

$$u(x) = 0 \tag{1}$$

is to embed (1) in a continuum of problems

$$H(t, x) = 0 \ (0 \le t \le 1), \tag{2}$$

which can easily be resolved when $t = 0$. When $t = 1$, problem (2) becomes (1). In the case when it is possible to continue the solution for all t in $[0, 1]$ then (1) is solved. This method is called continuation with respect to a parameter [1–25].

In this paper, sufficient conditions are given in order to prove that a compact C^1-mapping $f : X \to Z$ has a fixed point, where X, Y are Hilbert spaces, "$X \subseteq Y$" a continuous embedding and $Z = i(X)$ provided with the norm of Y. Other conditions, sufficient to guarantee the existence of fixed points, have been given by the author in a finite-dimensional setting [8–15,17,18,20,24,25], in an infinite-dimensional setting for Fredholm mappings [16,21], and in compact Banach manifolds modelled on $\mathbb{R}^n$ [23]. Continuation methods were used in the proofs of these papers and to prove the existence of open trajectories and lower bound of stable stationary trajectories of ordinary differential equations [19,22]. Here we also use continuation methods. The proof supplies the existence of implicitly defined mappings whose ranges reach fixed points [1–4,6–25]. The key is the use of the Surjective Implicit Function Theorem [27] and the properties of continuous embeddings (see [26] and [27]).

We briefly recall some theorems and concepts to be used.

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Definitions ([26,27]). Henceforth we will assume that X and Y are Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Mapping $F : X \to Y$ is called *weakly coercive* if and only if $\|F(x)\| \to \infty$ as $\|x\| \to \infty$.

Mapping $F : D(F) \subseteq X \to Y$, is said to be *compact* whenever it is continuous and the image $F(B)$ is relatively compact (i.e. its closure $\overline{F(B)}$ is compact in Y for every bounded subset $B \subset D(F)$.

Mapping F is said to be *proper* whenever the pre-image $F^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(F)$.

X, Y are called *topologically isomorphic* if and only if there is a linear homeomorphism (*topological isomorphism*) $L : X \to Y$.

Let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L : X \to Y$.

Let $\mathrm{Isom}(X, Y)$ denote the set of all the topological isomorphisms $L : X \to Y$.

Let $f : X \to Y$ be a C^1-mapping, where X and Y are Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The point $x \in X$ is called a *regular point* of f if and only if $f'(x)$ is surjective and the null space $\ker(f'(x))$ *splits* X into a topological direct sum. A point $y \in Y$ is called a *regular value* of f if and only if the set $f^{-1}(y)$ is empty or consists solely of regular points.

Let X and Y be normed spaces over $\mathbb{K}$. We say that the embedding "$X \subseteq Y$" is continuous if and only if there exists an operator $j : X \to Y$ which is linear, continuous, and injective. In terms of sequences, the embedding "$X \subseteq Y$" is continuous if and only if, when $n \to \infty$

$$x_n \to x \text{ in } X \text{ implies } j(x_n) \to j(x) \text{ in } Y.$$

Theorem 1 (*The Surjective Implicit Function Theorem. ([27], pp. 268–269)). Let X, Y, Z be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let*

$$F : U(u_0, v_0) \subseteq X \times Y \to Z$$

be a C^1-mapping on an open neighbourhood of the point (u_0, v_0). Suppose that:

(i) $F(u_0, v_0) = 0$, *and*
(ii) $F_v(u_0, v_0) : Y \to Z$ *is surjective.*

Therefore the following are true:

(a) *Let $r > 0$. There is a number $\rho > 0$ such that, for each given $u \in X$ with $\| u - u_0 \| < \rho$, the equation*

$$F(u, v) = 0$$

has a solution v, denoted by $v(u)$, such that $\| v - v_0 \| < r$. In particular, the limit $u \to u_0$ in X implies $v(u) \to v_0$.

(b) *There is a number $d > 0$ such that $\| v(u) \| \leq d \| F_v(u_0, v_0)v(u) \|$.*

2. Continuous embeddings and continuation method

Theorem. *Let "$X \subseteq Y$" be a continuous embedding, where X $(X, \| \cdot \|_1)$ and Y $(Y, \| \cdot \|_2)$ are Hilbert spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and*

$$i : X \to Y$$

is the injective, linear continuous mapping, and where $i(X) := Z$ is provided with $\| \cdot \|_2$. Let us suppose the following hold:

(i) *Mapping $f : X \to Z$ is a C^1-compact mapping and mapping $tf(x) - i(x)$ is weakly coercive.*
(ii) $\overline{i(X)} = i(X)$.
(iii) *For any fixed t, which belongs to $[0, 1]$, the zero is a regular value of the mapping $tf(x) - i(x)$.*

Therefore f has a fixed point x^*, i.e., $f(x^*) = x^*$.

Proof. (a) Since i is a linear mapping then $i(X) = Z$ is a linear subspace of Y, which is closed by hypothesis (ii), and since $(Y, \| \cdot \|_2)$ is a Hilbert space, then Proposition 5 in [5] (p. 71) implies $(Z, \| \cdot \|_2)$ is also a Hilbert space. Since i is a bijective linear continuous mapping between $(X, \| \cdot \|_1)$ and $(Z, \| \cdot \|_2)$, which are Banach spaces, then Proposition 1 in [27] (pp. 179–180) implies i is a topologic isomorphism between these Banach spaces.

(b) Let us construct the mapping

$$H : X \times [0, 1] \to Z, \quad \text{where } H(x, t) := tf(x) - i(x).$$

We will prove here that H is a proper mapping, and since $0 \in Z$ is a compact set then $H^{-1}(0)$ is also a compact set.

Let C be any fixed compact subset of Z, and let any sequence be fixed such as $(H(x_n, t_n))_{n \geq 1}$ which belongs to C. It suffices to show that the sequence $((x_n, t_n))_{n \geq 1}$ contains a convergent subsequence $((x_{n''}, t_{n''}))_{n'' \geq 1}$. Since

$$(x_{n''}, t_{n''}) \to (u, t) \quad \text{as } n'' \to \infty,$$

and H is continuous, $H(u, t) \in C$, that is, $(u, t) \in H^{-1}(C)$.