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Generalized 2-KKM theorems and their applications in hyperconvex metric spaces

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Abstract

In this work, we first define the 2-KKM mapping and the generalized 2-KKM mapping on a metric space, and then we apply the property of the hyperconvex metric space to get a KKM theorem and a fixed point theorem without a compactness assumption. Next, by using this KKM theorem, we get some variational inequality theorems and minimax inequality theorems.

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1. Introduction and preliminaries

In 1929, Knaster, Kurnatoaski and Mazurkiewicz [5] proved the well-known KKM theorem on the n-simplex. In 1961, Ky Fan [2] generalized the KKM theorem to the infinite dimensional topological vector space. Later, the KKM theorem and related topics, for example, the matching theorem, fixed point theorem, coincidence theorem, variational inequalities, minimax inequalities and so on, were presented in papers by a number of authors. The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [1] who proved that a hyperconvex space is an absolute retract. In 1996 Khamsi [3] established an analogue of the famous KKM maps principle due to Ky Fan for hyperconvex metric spaces, while, in [4], Kirk et al. proved a KKM theorem for the generalized KKM mappings on hyperconvex metric spaces, obtaining fixed point theorems, Ky Fan matching theorems and minimax inequalities under compactness assumptions.

In this work, we first define the 2-KKM mapping and the generalized 2-KKM mapping on a metric space, and then we apply the property of the hyperconvex metric space to get a KKM theorem and a fixed point theorem without a compactness assumption. Next, by using this KKM theorem, we get some variational inequality theorems and minimax inequality theorems.

Let X and Y be two sets, 2^X denote the class of all nonempty subsets of X, and $T : X \to 2^Y$ be a set-valued mapping. We shall use the following notation in the sequel.

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(i) $T(x) = \{y \in Y : y \in T(x)\}$,
(ii) $T(A) = \cup_{x \in A} T(x)$,
(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,
(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$,
(v) the set-valued mapping $T^c : X \to 2^Y$ is defined by $T^c(x) = Y \setminus T(x)$, for $x \in X$, and
(vi) the set-valued mapping $T^* : Y \to 2^X$ is defined by $T^*(y) = X \setminus T^{-1}(y)$, for $y \in Y$.

Definition 1. Let (M, d) be a metric space. A subset X of M is called admissible if it is an intersection of closed balls in M. The collection of all admissible subsets in M is denoted by $\mathcal{A}(M)$. The smallest admissible set containing a bounded subset X of M is called the admissible hull of X and denoted by $\mathrm{ad}(X)$. So

$$\mathrm{ad}(X) = \bigcap_{x \in M} B(x, r_x(X)),$$

where $B(x, r_x(X))$ is the closed ball centered at x with radius $r_x(X) \geq 0$ and $r_x(X) = \sup\{d(x, y) : y \in X\}$.

Remark 1. Let X be a subset of a metric space (M, d). Then X is called subadmissible if for each $A \in \langle X \rangle$, $\mathrm{ad}(A) \subset X$. Obviously, if X is an admissible subset of M, then X must be subadmissible, but the converse need not be true.

Definition 2. A metric space (M, d) is called hyperconvex if for any collection of points $\{x_\alpha\}$ of X and for r_α a collection of non-negative reals such that

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta,$$

then

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \phi.$$

Definition 3. Let M be a metric space and $X \subset M$. A set-valued mapping $F : X \to 2^M$ is called a KKM map if

$$\mathrm{ad}(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i),$$

for any $x_1, x_2, \ldots, x_n \in X$.

Definition 4. Let M be a metric space and $X \subset M$. A set-valued mapping $F : X \to 2^M$ is called a 2-KKM map if for each $x_1, x_2 \in X$,

$$x_1 \in F(x_1), \qquad x_2 \in F(x_2) \quad \text{and} \quad \mathrm{ad}(\{x_1, x_2\}) \subset F(x_1) \cup F(x_2).$$

Remark 2. If $F : X \to 2^X$ is a KKM map, then F is a 2-KKM map, but the converse need not hold.

For an counterexample, let $(M, d) = (\Re^2, \|.\|_2)$, $x_1 = (-1, 0)$, $x_2 = (1, 0)$, $x_3 = (0, 3^{1/2})$, and let $F : M \to 2^M$ be defined by

$$F(x) = \begin{cases} B(x, 1) & x \in \{x_1, x_2, x_3\}, \\ M & x \in \Re^2 \setminus \{x_1, x_2, x_3\}. \end{cases}$$

Then F is a 2-KKM mapping, but F is not a KKM mapping.

Definition 5. Let X be a nonempty set, and let Y be a nonempty subset of a metric space (M, d). A set-valued mapping $G : X \to 2^Y$ is called a generalized KKM mapping if for each $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$, there exists $\{y_1, y_2, \ldots, y_n\} \in \langle Y \rangle$ such that for each $\{i_1, i_2 \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $\mathrm{ad}\{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\} \subset \cup_{j=1}^{k} G(x_{i_j})$.

Definition 6. Let X be a nonempty set and Y a metric space. A set-valued mapping $F : X \to 2^Y$ is called a generalized 2-KKM map if for each $x_1, x_2 \in X$, there exists $y_1, y_2 \in Y$ such that

$$y_1 \in F(x_1), \qquad y_2 \in F(x_2) \quad \text{and} \quad \mathrm{ad}(\{y_1, y_2\}) \subset F(x_1) \cup F(x_2).$$