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Existence of positive almost automorphic solutions to neutral nonlinear integral equations[☆]

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Abstract

This paper is concerned with neutral nonlinear delay integral equations. We establish an existence theorem for positive almost automorphic solutions to the equations, which extend some existing results even in the case of almost periodicity. Some examples are given to illustrate our results.

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1. Introduction

In [11], Fink and Gatica considered the existence of a positive almost periodic solution for the following delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s)) \mathrm{d}s, \tag{1.1}$$

which is a model for the spread of a disease (cf. [6]). Since then, the existence of positive almost periodic solutions for various forms of equation (1.1) has been discussed by many researchers (see, e.g., [2,9,17] and references therein). Recently, Ait Dads and Ezzinbi [1] considered the existence of a positive almost periodic solution for the following neutral nonlinear delay integral equation:

$$x(t) = \gamma x(t-\tau) + (1-\gamma) \int_{t-\tau}^{t} f(s, x(s)) \mathrm{d}s, \tag{1.2}$$

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where $\gamma \in [0, 1)$ and $f(t, \cdot)$ is nondecreasing. In this work, we investigate the existence of a positive almost automorphic solution for the following more general equation:

$$x(t) = \gamma x(t-\tau) + (1-\gamma)\int_{t-\tau}^{t} \sum_{i=1}^{n} f_i(s, x(s)) g_i(s, x(s)) \mathrm{d}s, \tag{1.3}$$

where $f_i(t, \cdot)$ is nondecreasing and $g_i(t, \cdot)$ is nonincreasing.

The existence of periodic and almost periodic solutions for neutral integral equations is an interesting topic (see, e.g., [1,4,5] and references therein). On the other hand, the concept of almost automorphy, which was introduced by Bochner [3], is an important generalization of the classical almost periodicity. Recently, a lot of attention has paid to the existence of almost automorphic solutions to various equations (including evolution equations, functional differential equations, dynamical systems, etc.); see, e.g., [10,12,15,16,13].

However, as far as we know, there are few results available in the literature on positive almost automorphic solutions for Eq. (1.1) and (1.2) and its variants. In [8], we presented some sufficient conditions for the existence of positive almost automorphic solutions to a generalized version of (1.1). In this paper, a new existence theorem for positive almost automorphic solutions to (1.3) is obtained. As a corollary, we present some existence theorems for positive almost periodic solutions to (1.3), which generalize the corresponding result of [1] (see Remarks 3.6 and 4.3).

Throughout this paper, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, and by Ω a subset in $\mathbb{R}$. This paper is organized as follows. In Section 2, we recall some definitions and basic results. In Section 3, we present our main results and their proofs. In the last section, we give some examples.

2. Definitions and preliminaries

Let X be a real Banach space. A closed convex set P in X is called a convex cone if the following conditions are satisfied:

(i) If $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$.
(ii) If $x \in P$ and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\leq$ in X through

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

A cone P is called normal if there exists a constant $k > 0$ such that

$$0 \leq x \leq y \quad \text{implies that } \|x\| \leq k\|y\|,$$

where $\|\cdot\|$ is the norm on X. We denote by P^o the interior set of P. A cone P is called a solid cone if $P^o \neq \emptyset$.

Definition 2.1. Let X be a real Banach space and $E \subset X$. An operator $A : E \times E \to X$ is called a mixed monotone operator if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e. $x_i, y_i \in E$ $(i = 1, 2)$, $x_1 \leq x_2$ and $y_1 \geq y_2$ implies that $A(x_1, y_1) \leq A(x_2, y_2)$. An element $x^* \in E$ is called a fixed point of A if $A(x^*, x^*) = x^*$.

In the proof of our main results, we will need the following fixed point theorem.

Theorem 2.2. *Let P be a normal and solid cone in a real Banach space X. Suppose that the operator $A = B + D^* : P^o \times P^o \to P^o$ satisfies:*

(A1) *$B : P^o \times P^o \to P^o$ is a mixed monotone operator and there exist a constant $t_0 \in [0, 1)$ and a function $\phi : (0, 1) \times P^o \times P^o \to (0, +\infty)$ such that for each $x, y \in P^o$ and $t \in (t_0, 1)$, $\phi(t, x, y) > t$ and*

$$B(tx, t^{-1}y) \geq \phi(t, x, y) B(x, y);$$

(A2) *there exist $x_0, y_0 \in P^o$ such that $x_0 \leq y_0$, $x_0 \leq A(x_0, y_0)$, $A(y_0, x_0) \leq y_0$ and*

$$\inf_{x, y \in [x_0, y_0]} \phi(t, x, y) > t, \quad \forall t \in (t_0, 1);$$

(A3) *$D^*(x, y) := D(x)$ for any $(x, y) \in X \times X$, where $D : X \to X$ is a positive linear operator satisfying $D(P^o) \subset P^o \cup \{\theta\}$.*