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A new iteration process for finite families of generalized Lipschitz pseudo-contractive and generalized Lipschitz accretive mappings

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Abstract

A new iteration process is introduced and proved to converge strongly to a common fixed point for a finite family of generalized Lipschitz nonlinear mappings in a real reflexive Banach space E with a uniformly Gâteaux differentiable norm if at least one member of the family is pseudo-contractive. It is also proved that a slight modification of the process converges to a common zero for a finite family of generalized Lipschitz accretive operators defined on E. Results for nonexpansive families are obtained as easy corollaries. Finally, the new iteration process and the method of proof are of independent interest.

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1. Introduction

Let E be a real Banach space with dual E^*. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f^*\| = \|x\|\},$$

where $\langle ., . \rangle$ denotes the pairing between elements of E and elements of E^*. It is well known that if E is smooth then J is single-valued. In the sequel, single-valued normalized duality mapping will be denoted by j.

A mapping $T : D(T) \subset E \to E$ with domain $D(T)$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in D(T)$. The mapping T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}_{n\ge1} \in [1, +\infty)$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all $x, y \in D(T)$; and it is said to be *Lipschitz* if there exists $L_1 > 0$ such that for all $x, y \in D(T)$, $\|Tx - Ty\| \le L_1 \|x - y\|$. The mapping T is called *pseudo-contractive* if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le \|x - y\|^2.$$

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An important class of operators closely related to the class of pseudo-contractive ones is that of accretive mappings. A mapping $A : D(A) \subset E \to E$ is said to be *accretive* if for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0.$$

It is easy to see that A is accretive if and only if $(I - A)$ is pseudo-contractive. The accretive operators were independently introduced by Browder [4] and Kato [23] in 1967. The importance of these operators is well known.

Iterative methods for approximating zeros of accretive operators or equivalently zeros of pseudo-contractive maps have been studied extensively by various authors (see e.g., [3,8,10,12,13,15,17,18,34]).

The main tool for approximation of fixed points of nonlinear mappings (when such fixed points exist) remains the iterative technique. A mapping $T : D(T) \subset E \to E$ is said to be *generalized Lipschitz* if there exists $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|x - y\|)$, for all $x, y \in D(T)$. Clearly, every Lipschitz map is generalized Lipschitz. Furthermore, every map with bounded range is also a generalized Lipschitz map. For an example which shows that the class of generalized Lipschitz maps *properly* includes the class of Lipschitz maps and those of mappings with bounded range, the reader may consult, for example, [7].

Markov [28] (see also Kakutani [22], 1938) showed that if a commutating *family* of bounded *linear* transformations $T_\alpha, \alpha \in \Delta$, ($\Delta$ an arbitrary index set) of a normed linear space E into itself leaves some nonempty compact convex subset K of E invariant, then the family has at least one common fixed point. (The actual result of Markov is more general than this but this version is adequate for our purposes here). Motivated by this result, De Marr [27] in 1963 studied the problem of the existence of a common fixed point for a family of *nonlinear* maps, and proved the following theorem.

Theorem DM (*[27]*). *Let E be a Banach space and K be a nonempty compact convex subset of E. If Ω is a nonempty commuting family of nonexpansive mappings of K into itself, then the family Ω has a common fixed point in K.*

Browder [4] proved the result of De Marr in a uniformly convex Banach space, requiring only that K be nonempty closed bounded and convex. For other fixed point theorems for families of nonexpansive mappings the reader may consult any of the following references: Belluce and Kirk [2]; Lim [24] and Bruck [5].

Within the past 10 years or so, considerable research efforts have been devoted to developing iterative methods for approximating *common fixed points* (assuming existence) for families of several classes of nonlinear mappings (see, e.g., [1,9,16,19–21,30,32,33,35]).

In this paper, which is basically a sequel to [14], we construct a new iterative sequence for the approximation of common fixed points of finite families of generalized Lipschitz pseudo-contractive and generalized Lipschitz accretive operators (assuming existence). Furthermore, our new iteration scheme and our methods of proof are of independent interest.

2. Preliminaries

Let K be a nonempty bounded, closed and convex subset of a Banach space E and let the diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (see e.g. [6,11,25]) is defined by $N(E) := \inf\{\frac{d(K)}{r(K)} : K$ is a closed convex and bounded subset of E with $d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [26]).

In the sequel, we shall make use of the following lemmas.

Lemma 1. *Let E be an arbitrary real Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.