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Limitations of frequency domain approximation for detecting chaos in fractional order systems

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Abstract

In this paper, we analytically study the influences of using frequency domain approximation in numerical simulations of fractional order systems. The number and location of equilibria, and also the stability of these points, are compared between the original system and its frequency based approximated counterpart. It is shown that the original system and its approximation are not necessarily equivalent according to the number, location and stability of the fixed points. This problem can cause erroneous results in special cases. For instance, to prove the existence of chaos in fractional order systems, numerical simulations have been largely based on frequency domain approximations, but in this paper we show that this method is not always reliable for detecting chaos. This approximation can numerically demonstrate chaos in the non-chaotic fractional order systems, or eliminate chaotic behavior from a chaotic fractional order system.

Keywords: Fractional order system; Chaos; Frequency domain approximation

1. Introduction

Fractional calculus as an extension of ordinary calculus has a 300-year-old history. It has been found that the behavior of many physical systems can be properly described by using the fractional order system theory. For example heat conduction [1], dielectric polarization [2], electrode–electrolyte polarization [3], electromagnetic waves [4], viscoelastic systems [5], quantum evolution of complex systems [6], quantitative finance [7] and diffusion waves [8] are known dynamical systems governed by the fractional order equations. In fact, real world processes generally or most likely are fractional order systems [9]. Furthermore, fractional order controllers such as the CRONE controller [10], TID controller [11], fractional PID controller [12] and lead–lag compensator [13] have already been implemented to improve the performance and robustness of closed loop control systems.

In recent years, considerable attention has been paid to finding chaotic behaviors in fractional order models. For example, it has been found that some fractional order differential systems such as the fractional order Chua circuit [14], the fractional order Duffing system [15], the fractional order jerk model [16], the fractional order Chen system [17], the fractional order Lü system [18], the fractional order Rössler system [19], the fractional order Arneodo system [20]

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and the fractional order Newton–Leipnik system [21] demonstrate chaotic behavior. In most of the literature in the fractional chaos field, the existence of chaotic behavior in fractional order systems has been validated only by numerical simulations. The methods used in numerical simulation of these chaotic fractional order systems are based on two different viewpoints. The first viewpoint is that of simulating a fractional system by numerically solving its fractional differential equations. Methods applied to find numerical solutions of differential equations directly approximate the response of a fractional order system and are called "time domain methods". One of the best methods in this category is the improved version of the Adams–Bashforth–Moulton algorithm and it is proposed on the basis of the predictor–corrector scheme. This method was introduced in [22,23] and has been used in some of the literature such as [21,24–26].

Another viewpoint as regards simulating a fractional order system is based on the frequency domain approximations of the fractional operators. From the control theory viewpoint, the methods proposed for obtaining an integer order continuous model, such as rational approximation of the irrational transfer function, can be divided into two groups: methods using continued fraction expansions (CFE) and interpolation techniques, and the methods using curve fitting or identification techniques [27]. For example, Carlson's method [28] and Matsuda's method [29] belong to the first group and Oustaloup's method [30,31] and Charef's method [32,33] are classified in the second group. To simulate a fractional order system by using the frequency domain approximations, the fractional order equation of the system is first considered in the frequency domain and then the Laplace form of the fractional integral operator is replaced by its integer order approximation (this approximation is often obtained by using methods from the second group such as Charef's method). Then the approximated equations in the frequency domain are transformed into the time domain. The resulting ordinary differential equations can be numerically solved by applying well known numerical methods. The "frequency domain methods" have been used in many papers (for example [14,16–20,34]) to simulate the behavior of chaotic fractional order systems.

This paper is organized as follows. Section 2 briefly describes basic concepts in fractional calculus, fractional systems, and the stability of these systems. We explain in Section 3 how the solution for a fractional order system is approximated using the approximation methods mentioned. In Section 4, the number and location of equilibria, and also the stability of these points, in the frequency based approximation model are compared with those for the original system. In Section 5, we show that using frequency domain approximation in simulation of a fractional order system may cause generation of fake chaotic behavior or may eliminate chaos from a fractional order chaotic system. Also, on the basis of the stability theorem in a fractional order system, the required condition by which a fractional order system demonstrates chaos and produces a one-scroll, two-scroll or multi-scroll chaotic attractor is discussed in this section. Conclusions in Section 6 close the paper.

2. Introduction to fractional calculus

The differintegral operator, denoted by ${}_aD_t^q$, is a combined differentiation and integration operator commonly used in fractional calculus. This operator represents taking both the fractional derivative and the fractional integral in a single expression and is defined by

$$ {}_aD_t^q = \begin{cases} \dfrac{\mathrm{d}^q}{\mathrm{d}t^q}, & q > 0 \\ 1, & q = 0 \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-q}, & q < 0. \end{cases} \tag{1} $$

There are different definitions for fractional derivatives [35]. The most commonly used definitions are the Grunwald–Letnikov, Riemann–Liouville and Caputo definitions. These definitions are briefly discussed on the following lines.

Grunwald–Letnikov definition:

$$ {}_aD_t^q f(t) = \frac{\mathrm{d}^q f(t)}{\mathrm{d}(t-a)^q} = \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f\left(t - j\left[\frac{t-a}{N}\right]\right). \tag{2} $$