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Boundary blow-up elliptic problems of Bieberbach and Rademacher type with nonlinear gradient terms[☆]

Zhijun Zhang

Department of Mathematics and Informational Science, Yantai University, Yantai, Shandong 264005, PR China

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Abstract

By a perturbation method and constructing comparison functions, we show the exact asymptotic behaviour of solutions near the boundary to nonlinear elliptic problems $\triangle u \pm |\nabla u|^q = b(x)e^u$, $x \in \Omega$, $u|_{\partial\Omega} = +\infty$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $q \geq 0$, b is non-negative and non-trivial in Ω, which may be vanishing on the boundary.

Keywords: Semilinear elliptic equations; Nonlinear gradient terms; Large solutions; The asymptotic behaviour

1. Introduction and the main results

The purpose of this paper is to investigate the exact asymptotic behaviour of the solutions near the boundary to the following model problem:

$$\triangle u \pm |\nabla u|^q = b(x)e^u, \quad x \in \Omega, \quad u|_{\partial\Omega} = +\infty, \qquad (P_\pm)$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \text{dist}(x, \partial\Omega) \to 0$, and the solution is called 'a large solution' or 'an explosive solution', Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $q \geq 0$, b satisfies

(b_1) $b \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, 1)$, and is non-negative in Ω;

(b_2) b has the property: if $x_0 \in \Omega$ and $b(x_0) = 0$, then there exists a domain Ω_0 such that $x_0 \in \Omega_0 \subset \Omega$ and $b(x) > 0$, $\forall x \in \partial\Omega_0$.

The main feature of this paper is the presence of the three terms: the nonlinear term e^u, the nonlinear gradient term $\pm|\nabla u|^q$ and the weight $b(x)$ which may be vanishing not only on large parts of Ω but also on the boundary and includes a large class of functions.

First, let's review the following model

$$\triangle u = b(x)e^u, \quad x \in \Omega, \quad u|_{\partial\Omega} = +\infty. \qquad (1.1)$$

For $b \equiv 1$ in Ω: the problem goes back to Bieberbach's pioneering work in 1916 and Rademacher's work in 1943 (see, for example, [18]) for $N = 2$ and $N = 3$. They showed that the problem (1.1) has one solution $u \in C^2(\Omega)$

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E-mail address: zhangzj@ytu.edu.cn.

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such that $|u(x) - \ln(d(x))^{-2}|$ is bounded on Ω. In this case, the problem arises in the study of an electric potential in a glowing hollow metal body, and plays an important role in the theory of Riemannian surfaces of constant negative curvatures and in the theory of automorphic functions. For general increasing nonlinearities $g(u)$ instead of e^u, Keller–Osserman [15,20] first supplied a necessary and sufficient condition $\int_1^\infty \frac{ds}{\sqrt{G(s)}} < \infty$ where $G'(s) = g(s)$ for the existence of large solutions to problem (1.1). Moreover, by the ordinary differential equation theory and the comparison principle, Lazer–McKenna [19] showed that problem (1.1) has a unique solution $u \in C^2(\Omega)$ and

$$u(x) - \ln 2(d(x))^{-2} \to 0 \quad \text{as } d(x) \to 0,$$

and, recently, Bandle [1] showed that

$$u(x) = \ln 2(d(x))^{-2} + (N-1)H(\bar{x})d(x) + o(d(x)) \quad \text{as } x \to \partial\Omega,$$

where $\partial\Omega \in C^4$ is compact and $\bar{x}$ is the nearest point to x on $\partial\Omega$, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at $\bar{x}$.

For $b > 0$ on $\bar{\Omega}$, Lazer–McKenna [18] showed that if Ω is a bounded domain in $\mathbb{R}^N$ which satisfies a uniform external sphere condition and $b \in C(\bar{\Omega})$, then there is at most one solution $u \in C^2(\Omega)$ to problem (1.1) and for any such solution $|u(x) + 2\ln d(x)|$ is bounded on Ω (Theorem 4.1). Moreover, if $b \in C^\alpha_{\text{loc}}(\Omega)$, and is bounded above, then there is at least one solution $u \in C^2(\Omega)$ to problem (1.1) (Theorem 4.2). They also gave a proof of uniqueness for a bounded domain Ω which is star-shaped with no smoothness assumption on $\partial\Omega$ (Theorem 3.1). For $b \in C^\alpha_{\text{loc}}(\Omega)$, $b > 0$ in Ω, García-Melián [9] showed that problem (1.1) has at least one solution $u \in C^2(\Omega)$ such that

$$-m - (2+\gamma_1)\ln(d(x)) \le u(x) \le M - (2+\gamma_2)\ln(d(x)), \quad \forall x \in \Omega,$$

provided that b satisfies the following assumptions: there exist constants $C_1, C_2 > 0$ and $\gamma_1 \ge \gamma_2 > -2$ such that

$$C_2(d(x))^{\gamma_2} \le k_1(x) \le C_1(d(x))^{\gamma_1}, \quad \forall x \in \Omega,$$

where m, M are positive constants. In particular, if $\gamma_1 = \gamma_2 = \gamma > -2$, then

$$\lim_{d(x)\to 0} \frac{u(x)}{-\ln(d(x))} = 2 + \gamma.$$

When b satisfies (b_1) and (b_2), Tao and the author [22,26] showed the existence of solutions to problem (1.1).

Now we introduce a class of functions.

Let Λ denote the set of all positive non-decreasing functions $k \in C^1(0, \nu)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{\int_0^t k(s)ds}{k(t)}\right) = l_k. \tag{1.2}$$

We note that for each $k \in \Lambda$,

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)ds}{k(t)} = 0 \quad \text{and} \quad l_k \in [0, 1].$$

The set Λ was first introduced by Cîrstea and Rădulescu [7].

Let b satisfy (b_1) and

(b_3) $\lim_{d(x)\to 0+} \frac{b(x)}{k^2(d(x))} = c_0 > 0$ for some $k \in \Lambda$,

more recently, by Karamata regular varying theory and the extreme value theory, Cîrstea [6] showed that if $l_k > 0$ and $c > 0$ then the following problem

$$\triangle u = b(x)\left(e^{cu} - 1\right), \quad u \ge 0, x \in \Omega, \quad u|_{\partial\Omega} = +\infty, \tag{1.3}$$

has a unique solution $u \in C^2(\Omega)$ satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{-\ln(d(x))} = \frac{2}{cl_k}.$$