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On almost smooth functions and piecewise smooth functions☆

Liqun Qi[a], Paul Tseng[b,*]

[a] Department of Mathematics, City University of Hong Kong, Kowloon Tong, Kowloon, Hong Kong
[b] Department of Mathematics, The University of Washington, Seattle, WA 98195-4350, USA

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Abstract

Piecewise smooth (PS) functions are perhaps the best-known examples of semismooth functions, which play key roles in the solution of nonsmooth equations and nonsmooth optimization. Recently, there have emerged other examples of semismooth functions, including the p-norm function ($1 < p < \infty$) defined on $\mathbf{R}^n$ with $n \geq 2$, NCP functions, smoothing/penalty functions, and integral functions. These semismooth functions share the special property that their smooth point sets are locally connected around their nonsmooth points. By extending a result of Rockafellar, we show that the smooth point set of a PS function cannot have such a property. This shows that the above functions, though semismooth, are not PS. We call such functions *almost smooth* (AS). We show that the B-subdifferential of an AS function at a point has either one or infinitely many elements, which contrasts with PS functions whose B-subdifferential at a point has only a finite number of elements. We derive other useful properties of AS functions and sufficient conditions for a function to be AS. These results are then applied to various smoothing/penalty functions and integral functions.

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1. Introduction

In the past two decades, nonsmooth functions have emerged to play important roles in optimization [10,15,21,42]. These functions, particularly those defined on finite-dimensional spaces, tend to be close to being smooth (i.e., continuously differentiable) in the sense that they are continuous everywhere and smooth almost everywhere. A well-known class of functions of this type are the *piecewise smooth* (*PS*) functions, which are locally representable as a selection from a finite collection of smooth functions; see Definition 1. These functions and their applications have been well studied [1,21–23,28,39,43]. Examples include the 1-norm function and the ∞-norm function and, more generally, piecewise linear functions composed with smooth functions. The PS functions have the nice properties

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* Corresponding author. Tel.: +1 206 543 1177; fax: +1 206 543 0397.
E-mail addresses: maqilq@cityu.edu.hk (L. Qi), tseng@math.washington.edu (P. Tseng).

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that they are *locally Lipschitzian* (i.e., *strictly continuous* [10,42]), and *semismooth* [25]. These properties ensure that these functions have generalized Jacobians [10] and B-subdifferentials [31]. Moreover, nonsmooth equations involving these functions can be efficiently solved by nonsmooth Newton methods [36]. Recently, there emerged many interesting nonsmooth functions that have similarly nice properties. These include the p-norm function on $\mathbf{R}^n$ with $1 < p < \infty$ and $n \geq 2$, the Fischer–Burmeister function [16] and other nonlinear complementarity (NCP) functions [15,20,24,32,45], smoothing/penalty functions [5–8,15,35], and integral functions involving splines and projection onto the nonnegative reals [12,13,18,34]. Are these functions PS also? If yes, then they can be treated within a well studied unifying framework. If not, then can they be treated systematically within some new framework?

By using an observation of Pang and Ralph [28] that the B-subdifferential of a PS function at a point has only finitely many distinct elements, Dontchev et al. [13] showed that the aforementioned integral functions are not PS due to their B-subdifferentials at the origin having infinitely many elements. This proof is specialized and does not appear to extend to other functions. This motivated a conjecture by the first author that a function defined on $\mathbf{R}^n$ with $n \geq 2$ is not PS if it is smooth everywhere except at one point. This conjecture was recently proved by Rockafellar [41], showing that the nonsmooth point set of a PS function defined on $\mathbf{R}^n$ ($n \geq 2$) cannot be a singleton (or, more generally, isolated points); see Theorem 1(c). Motivated by the above results, we make further studies in this paper of the nonsmooth point sets of PS functions, and use our results to formulate and study a new class of "nice" nonsmooth functions, which we call "almost smooth" functions, that are not PS but encompass the aforementioned functions like the p-norm function, smoothing/penalty functions, and integral functions.

Our first contribution is a generalization of the result of Rockafellar [41]. We show that the smooth point set of a PS function f defined on $\mathbf{R}^n$ ($n \geq 2$) is locally disconnected around each point where f is not *strictly differentiable* [42]; see Theorem 2(c). We also give a characterization of strict differentiability for PS functions. A corollary of this result is that the smooth point sets of f cannot be locally connected around all the nonsmooth points; see Corollary 1. Intuitively, the nonsmooth point set of a PS function f defined on $\mathbf{R}^n$ partitions $\mathbf{R}^n$ into multiple connected components, with f being smooth on the interior of each component. For example, when $n = 2$, the nonsmooth point set can be the union of lines and curves. All such lines and curves should either exclude their endpoints or extend to "infinity" without endpoints.

Our second contribution is a systematic treatment of nonsmooth functions such as the aforementioned p-norm function ($1 < p < \infty, n \geq 2$), NCP functions, smoothing/penalty functions, and integral functions, within a common framework. In particular, these functions are not only locally Lipschitzian and semismooth, they share the additional property that their smooth point sets are locally connected around all the nonsmooth points. Thus these functions are *not* PS. We call a function *weakly almost smooth* if it is locally Lipschitzian and has this additional property; see Definition 3. We call it *almost smooth* (respectively, *strongly almost smooth*) if, in addition, it is semismooth (respectively, strongly semismooth); see Definition 4. In what follows, we will often abbreviate "almost smooth" as "AS". We study the subdifferential properties of AS functions. In particular, we show that the B-subdifferential of a weakly AS function at a point contains either a single element (in this case it is strictly differentiable at that point) or infinitely many elements; see Theorem 3. This property further distinguishes AS functions from PS functions. We define the *principal part* of the B-subdifferential of a locally Lipschitzian function, a notion also used by Klatte and Kummer [21, Eq. (6.30)] in analyzing nonsmooth Newton methods. For PS functions, the B-subdifferential coincides with its principal part. We show that this is also true for a weakly AS function that is smooth everywhere except at isolated points; see Theorem 4. We also show that if f is smooth everywhere except at a point and f is positively homogeneous about that point, then f is AS. If in addition ∇f is locally Lipschitzian everywhere except at that point, then f is strongly AS; see Theorem 5. This provides an easy check for positively homogeneous functions to be AS.

In Section 4, we make further studies of AS functions, with more examples and applications. In Sections 4.1 and 4.2, the AS functions include the p-norm function defined on $\mathbf{R}^n$ ($1 < p < \infty$, $n \geq 2$), certain NCP functions, and integral functions associated with convex best interpolation. In Section 4.3, we derive general conditions for a class of smoothing/penalty functions for NCP and constrained optimization to be AS. We then apply them to various examples, including the Chen–Mangasarian class of smoothing functions and the exponential penalty function, to show they are AS. In Section 4.4, we study conditions for a certain integral function to be strongly AS. While particular instances of AS functions have been studied in disparate contexts [13,16,30,32,35], this is the first systematic study of such functions.

Throughout this paper, for any $x \in \mathbf{R}^n$, we use $\|x\|$ to denote the Euclidean norm of x. For any $\bar{x} \in \mathbf{R}^n$ and $\varepsilon > 0$, we denote the open Euclidean ball $\mathbf{B}_\varepsilon(\bar{x}) = \{x \in \mathbf{R}^n : \|x - \bar{x}\| < \varepsilon\}$. We use ":=" to mean "define". $\mathbf{R}_+$ and $\mathbf{R}_-$