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Existence and asymptotic expansion for a viscoelastic problem with a mixed nonhomogeneous condition

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Abstract

We study the initial-boundary value problem for a nonlinear wave equation given by

$$\begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)\mathrm{d}s + K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t = f(x,t), \quad 0 < x < 1, 0 < t < T, \\ u_x(0,t) = u(0,t), \quad u_x(1,t) + \eta u(1,t) = g(t), \\ u(x,0) = \widetilde{u}_0(x), \quad u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{1}$$

where $\eta \geq 0$; $p \geq 2$, $q \geq 2$; K, λ are given constants and $\widetilde{u}_0$, $\widetilde{u}_1$, f, g, k are given functions. In this paper, we consider three main parts. In Part 1 we prove a theorem of existence and uniqueness of a weak solution u of problem (1). The proof is based on a Faedo–Galerkin method associated with a priori estimates, weak convergence and compactness techniques. Part 2 is devoted to the study of the asymptotic behavior of the solution u as $\eta \to 0_+$. Finally, in Part 3 we obtain an asymptotic expansion of the solution u of the problem (1) up to order $N+1$ in three small parameters K, λ, η.

MSC: 35L20; 35L70

Keywords: Faedo–Galerkin method; Existence and uniqueness of a weak solution; Energy-type estimates; Compactness; Asymptotic expansion

1. Introduction

In this paper we will consider the following initial and boundary value problem:

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)\mathrm{d}s + F(u,u_t) = f(x,t), \quad 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u_x(0,t) = \eta_0 u(0,t), \qquad u_x(1,t) + \eta u(1,t) = g(t), \tag{1.2}$$

$$u(x,0) = \widetilde{u}_0(x), \qquad u_t(x,0) = \widetilde{u}_1(x), \tag{1.3}$$

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where $F(u, u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, with $\eta \geq 0$, $\eta_0 > 0$; $p \geq 2$, $q \geq 2$; K, λ are given constants and $\widetilde{u}_0$, $\widetilde{u}_1$, f, g, k are given functions satisfying conditions specified later.

In a recent paper, [1], Berrimia and Messaoudi considered the problem

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)\mathrm{d}s = |u|^{p-2}u, \quad x \in \Omega, t > 0, \tag{1.4}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.5}$$

$$u(x,0) = \widetilde{u}_0(x), \qquad u_t(x,0) = \widetilde{u}_1(x), \quad x \in \Omega, \tag{1.6}$$

where $p > 2$ is a constant, k is a given positive function, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. This type of problems have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established. In this regard, Cavalcanti et al. [3] studied the following equation

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)\mathrm{d}s + |u|^{p-2}u + a(x)u_t = 0, \quad \text{in } \Omega \times (0,\infty), \tag{1.7}$$

for $a : \Omega \to \mathbb{R}_+$, a function, which may be null on a part of the domain Ω. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\zeta_1 k(t) \leq k'(t) \leq -\zeta_2 k(t), \quad t \geq 0, \tag{1.8}$$

the authors established an exponential rate of decay.

In [5,6], Long and Alain Pham have studied problem (1.1) and (1.3) with $k \equiv 0$, $f(x,t) = 0$.

In [5], we have considered it with the mixed nonhomogeneous condition

$$u_x(0,t) = hu(0,t) + g(t), \quad u(1,t) = 0, \tag{1.9}$$

where $h > 0$ is a given constant; in [6] with the more generalized boundary condition

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t H(t-s)u(0,s)\mathrm{d}s, \quad u(1,t) = 0. \tag{1.10}$$

In [7], Long and Diem have studied problem (1.1) and (1.3) with $k \equiv 0$, and the mixed homogeneous condition

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.11}$$

where h_0, h_1 are given non-negative constants with $h_0 + h_1 > 0$ and a right-hand side of the form

$$F = F(x,t,u,u_x,u_t). \tag{1.12}$$

In [2] Bergounioux et al. studied problem (1.1) and (1.3) with $k \equiv 0$, $F(u,u_t) = Ku + \lambda u_t$, and the mixed boundary conditions (1.2) standing for

$$u_x(0,t) = g(t) + hu(0,t) - \int_0^t H(t-s)u(0,s)\mathrm{d}s, \tag{1.13}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.14}$$

where $h \geq 0$, K, λ, K_1, λ_1 are given constants and g, H are given functions.

In [9], Long et al. obtained the unique existence, regularity and asymptotic expansion of the problem (1.1), (1.3), (1.13) and (1.14) in the case of $k \equiv 0$, $F(u,u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, with $p \geq 2$, $q \geq 2$; K, λ are given constants.

In [10], Long et al. gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)–(1.3) when $F(u,u_t) = Ku + \lambda u_t$, $\widetilde{u}_0 \in H^2$ and $\widetilde{u}_1 \in H^1$. In this case, the problem (1.1)–(1.3) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

In this paper, we consider three main parts. In Part 1, under conditions $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$; f, $f_t \in L^2(Q_T)$, $k \in W^{2,1}(0,T)$, $g \in H^2(0,T)$; $K, \lambda, \eta \geq 0$, $\eta_0 > 0$; $p, q \geq 2$, we prove a theorem of existence and uniqueness