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Hamiltonian-minimal Lagrangian submanifolds in Kaehler manifolds with symmetries

Yuxin Dong*

Institute of Mathematics, Fudan University, Shanghai, 200433, PR China
Key Laboratory of Mathematics for Nonlinear Sciences (Fudan University), Ministry of Education, PR China

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Abstract

By making use of the symplectic reduction and the cohomogeneity method, we give a general method for constructing Hamiltonian minimal Lagrangian submanifolds in Kaehler manifolds with symmetries. As applications, we construct infinitely many nontrivial complete Hamiltonian minimal Lagrangian submanifolds in CP^n and C^n.

MSC: 53C42; 53D12

1. Introduction

Let (M^{2m}, ω) be a symplectic manifold with a Riemannian metric g and let L be a Lagrangian submanifold in M. A normal vector field V along L is called a Hamiltonian variation if the one form $\alpha_V := \omega(V, \cdot)$ is exact. According to [15,16], the Lagrangian submanifold L is called Hamiltonian minimal if it is a critical point of the volume functional with respect to all Hamiltonian variations along L. In particular, this makes sense if M is a Kaehler manifold. A Hamiltonian minimal Lagrangian submanifold will be simply called H-minimal.

Proposition 1.1 (*[16]*). *Let (M, ω, g) be a Kaehler manifold. A Lagrangian submanifold $L \subset M$ is H-minimal if and only if its mean curvature vector H satisfies*

$$\delta\alpha_H = 0 \tag{1}$$

on L, where δ is the Hodge-dual operator of d on L.

H-minimal Lagrangian submanifolds offer a nice generalization of the minimal submanifold theory. It was Oh who first investigated these submanifolds (see [15,16]). One motivation to study them is its similarity to some models in incompressible elasticity [20,7]. In [16], the author commented that H-minimal Lagrangian submanifolds seem to exist more often than minimal Lagrangian submanifolds do. In [2], Castro and Urbano constructed some

* Corresponding address: Fudan University, Department of Mathematics, Math. Institute Fudan University, 200433, Shanghai, PR China. Tel.: +86 2165643047.
E-mail address: yxdong@fudan.edu.cn.

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exotic Hamiltonian tori in C^2. Afterwards, Helein and Romon constructed H-minimal surfaces via integrable system methods [7,8]. In [3,4], the authors constructed some H-minimal submanifolds in complex space forms of higher dimensions. Besides these explicit instances, Schoen and Wolfson [17] established some important existence and regularity results for two-dimensional H-minimal surfaces. On the other hand, Amarzaya and Ohnita study in [1] compact Hamiltonian stable minimal Lagrangian submanifolds with parallel second fundamental form in CP^n.

The aim of this paper is to give a general method for constructing H-minimal Lagrangian submanifolds in Kaehler manifolds with symmetries by symplectic reductions and the cohomogeneity method. Note that Eq. (1) is a third order PDE, which is more complicated than the usual minimal submanifold equation. Even for the usual minimal submanifold, the existence is a difficult area of study, due to the nonlinearity of the equation. Recently, the symmetry reduction method has led to some important progress in explicit construction of special Lagrangian submanifolds by several authors (see [11,12] and the references contained therein). In this paper, we will solve (1) by the same trick. Let G be a compact connected Lie group of holomorphic isometries of a Kaehler manifold M and let μ be the moment map of the G-action. First, we show that a G-invariant Lagrangian submanifold is H-minimal if and only if it is stationary with respect to any G-invariant Hamiltonian variation. From [12], we know that a G-invariant Lagrangian submanifold is contained in a level set of μ. The well-known Noether theorem tells us that the moment map μ is a conserved quantity for every G-invariant Hamiltonian deformation. This allows us to restrict the variational problem to a level set of μ. By combining symplectic reduction and the cohomogeneity method developed in [6], we can reduce Eq. (1) to a PDE on the symplectic quotient with the Hsiang–Lawson metric. We have a very nice correspondence between the G-invariant H-minimal Lagrangian submanifolds in M and the H-minimal Lagrangian submanifolds in the quotient space (see Corollary 2.8 and Theorem 2.9). The reduction procedure simplifies the original equation greatly. Actually, the reduced system becomes an ODE if the G-action is of cohomogeneity one. To demonstrate the procedure, we consider some concrete G-actions of cohomogeneity one on CP^n and C^n respectively. By solving the corresponding ODE, we construct infinitely many non-trivial closed H-minimal Lagrangian submanifolds and also non-trivial complete H-minimal Lagrangian submanifolds in CP^n and C^n. Here the H-minimal Lagrangian submanifolds are called nontrivial, if they are not minimal in the usual sense.

2. Symmetry reduction

Let M be a connected manifold with a differentiable G-action, where G is a compact, connected Lie group. For each $x \in M$ let G_x be the isotropy subgroup of x, and $G(x) \approx G/G_x$ be the orbit of x under G. Two orbits, $G(x)$ and $G(y)$, are said to be of the same type if G_x and G_y are conjugate in G. The conjugacy classes of the subgroup $\{G_x : x \in M\}$ are called the orbit types of the G-space M. The orbit types may be partially ordered as follows:

$$(H) \succ (K) \Longleftrightarrow \exists g \in G \quad \text{s.t. } K \supseteq gHg^{-1}$$

where (H) denotes the conjugacy class of H. We need the following important result [13]:

Proposition 2.1 (*Principal Orbit Type*). *Let M be a connected manifold with a differentiable G-action. Then there exists a unique orbit type (H) such that $(H) \succ (K)$ for all orbit types (K) of the action. Moreover, the union of all orbits of type (H), namely $M^* = \{x \in M : G_x \in (H)\}$, is an open, dense submanifold of M.*

Following [14] we call (H) in Proposition 2.1 the principal orbit type of the G-space M. If $(H') \neq (H)$ but $\dim H' = \dim H$, then (H') will be called an exceptional orbit type. All other orbit types will be called singular.

From now on we assume that M is a Kaehler manifold with Kaehler form ω and complex structure J. Let G be a compact, connected Lie group of holomorphic isometries of M. Let g be the Lie algebra of G, and g^* the dual space of g. Then a moment map for the G-action is a smooth map $\mu : M \to g^*$ such that

(a) $(\mathrm{d}\mu, \xi) = i_{\phi(\xi)}\omega$ for all $\xi \in g$, where $(,)$ denotes the pairing between g and g^*, and $\phi : g \to C^\infty(TM)$ is the infinitesimal action;
(b) $\mu(kx) = \mathrm{Ad}_k^*\mu(x), \forall k \in G$ and $x \in M$;

where Ad^* denotes the coadjoint action.

Now let G be an action on (M, ω, J) with its moment map μ. Let $Z(g^*)$ be the centre of g^*, i.e., the vector subspace of g^* fixed by the coadjoint action of G. If $c \in Z(g^*)$, we see from (b) that G induces an action on the level