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On positive solution for a class of degenerate quasilinear elliptic positone/semipositone systems

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Abstract

This paper deals with the existence and nonexistence of positive weak solutions of degenerate quasilinear elliptic systems with subcritical and critical exponents. The nonlinearities involved have semipositone and positone structures and the existence results are obtained by applying the lower and upper-solution method and variational techniques.

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1. Introduction

Castro, Hassanpour, and Shivaji in [1], and then different authors, focused their attention on a class of problems, the so-called semipositione problems, of the form

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a positive parameter, and $f : [0,\infty) \to \mathbb{R}$ is a monotone and continuous function satisfying the conditions

$$f(0) < 0, \tag{f_0}$$

$$\lim_{s\to\infty} f(s) = +\infty, \tag{f_1}$$

and also the sublinear condition at infinity, namely, $\lim_{s\to\infty} f(s)/s = 0$.

This kind of problem was motivated by the paper of Keller–Cohen in [2], where they studied a positone problem, which means the Dirichlet problem, involving as nonlinearity a positive and monotone function.

The semipositone problems are mathematically a challenging area in the study of positive solution of the Dirichlet problem. We refer the reader to the survey paper [3] (and [4] for $p \neq 2$) as well as to [5] and references therein.

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Actually, the authors in [5] were able to treat not only the sublinear case, but also the superlinear case by applying variational techniques.

Chhetri, Hai, and Shivaji [6], by using degree theory, studied the semipositone systems involving p-laplacian operator with $p > 1$, of the type

$$(P_p):\begin{cases}-\Delta_p u = \lambda f_1(v) & \text{in } \Omega,\\ -\Delta_p v = \lambda f_2(u) & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter, and $f_1, f_2 : [0,\infty) \to \mathbb{R}$ are monotone and continuous functions satisfying conditions (f_0), (f_1), and

$$\lim_{s\to\infty} \frac{\max\{f_1(s), f_2(s)\}}{s^{p-1}} = 0. \qquad (f_2)$$

While in [7], Hai and Shivaji proved an existence result for system (P_p) with the condition

$$\lim_{s\to\infty} \frac{f_1(M(f_2(s))^{1/(p-1)})}{s^{p-1}} = 0, \quad \text{for all } M > 0, \qquad (f_3)$$

instead of the condition (f_2) mentioned above. They applied the lower and upper solution method. We would like to mention that in both the papers mentioned above, only the autonomous case was considered.

On the other hand, in the scalar case, García and Peral in [8] proved that the problem involving p-laplacian operator with $1 < p < N$, of the type

$$\begin{cases}-\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega\end{cases}$$

does not have any nontrivial weak solution for all $\lambda > 0$ small enough. While the perturbed problem involving p-laplacian operator with $1 < p < N$

$$\begin{cases}-\Delta_p u = \lambda |u|^{p-2}u + |u|^{q-2}u & \text{in } \Omega,\\ u \geq 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega\end{cases}$$

has a nontrivial weak solution for all $\lambda > 0$ sufficiently small and $1 < q < p^*$ (see also Ghoussoub and Yuan [9]). Here $p^* = Np/(N-p)$ denotes the critical Sobolev exponent. Still in the scalar case, an important situation happens when $q = p^*$; in this case the inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is continuous, but not compact any longer. However, in a pioneering paper by Brezis and Nirenberg [10] the case $p = 2$ is treated. This type of problem involving the lack of compactness has been studied by many authors, and we would like to mention some of them, for instance, [8,9,11] and references therein. Recently, Adriouch and Hamidi [12] studied the system

$$\begin{cases}-\Delta_p u = \lambda |u|^{p_1-2}u + (\alpha+1)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega,\\ -\Delta_q u = \lambda |v|^{q-2}v + (\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega,\end{cases}$$

with Dirichlet or mixed boundary conditions, and supposing that the exponents verify $(\alpha+1)/p^* + (\beta+1)/q^* < 1$, and $1 < p_1 < p$. In [13], they handled the critical case $(\alpha+1)/p^* + (\beta+1)/q^* = 1$, with $p = q$. For the systems involving p-Laplacian or Laplacian operators, we would like to mention the papers, e.g., [14–16] and a survey paper [17].

The aim of this work is to extend or complement some of the above results for the quasilinear elliptic systems involving singularities of the type

$$\begin{cases}-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = \lambda g_1(x,u,v) & \text{in } \Omega,\\ -\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v) = \lambda g_2(x,u,v) & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega,\end{cases} \qquad (1)$$