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Asymptotic speed of propagation and traveling wave solutions for a lattice integral equation☆

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Abstract

We investigate the asymptotic speed of propagation and monotone traveling wave solutions for a lattice integral equation which is an epidemic model while the population is distributed on one-dimensional lattice $\mathbb{Z}$. It is proved that the asymptotic speed of propagation c_* coincides with the minimal wave speed.

MSC: 45G15; 92D30

Keywords: Asymptotic speed of propagation; Traveling wave solution; Lattice integral equation

1. Introduction

One of the epidemic models is the following integral equation

$$u(t,x)=\int_0^t A(t-\tau)\int_{R^n} g(u(\tau,\xi))V(x-\xi)\mathrm{d}\xi\mathrm{d}\tau+f(t,x),\quad t\geq 0, x\in\mathbb{R}^n. \tag{1.1}$$

The behaviors of (1.1) associated with a time-translation invariant homogeneous equation

$$u(t,x)=\int_{-\infty}^t A(t-\tau)\int_{\mathbb{R}} g(u(\tau,\xi))V(x-\xi)\mathrm{d}\xi\mathrm{d}\tau,\quad t\in\mathbb{R}, x\in\mathbb{R}^n, \tag{1.2}$$

were investigated by Diekmann [5–7], such as the existence of traveling wave solutions, the minimal wave speed of (1.1), the asymptotic speed of propagation of (1.1) etc. The problem of traveling wave solutions has been widely investigated for reaction-diffusion equations ([3,4,8,9,11,12] and [17]) and integral equations [13,14]. The concept of asymptotic speed of propagation was introduced by Aronson and Weinberger [1,15] for reaction-diffusion equations and applied by Aronson [2], Diekeman [5], Thieme [13], Thieme and Zhao [14], Radcliffe and Rass [10] to integro-differential and integral equations, and by Weng, Huang and Wu [16] to a lattice differential equation. The extensive references could be found in Thieme and Zhao [14].

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If one considers a discrete spatial variable $k \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, instead of $x \in \mathbb{R}^n$, that is, the population of the species is distributed on a one-dimensional lattice, one could have a model with the form

$$u_k(t) = \int_0^t \sum_{j\in\mathbb{Z}} A_{k-j}(t-\tau) g(u_j(\tau))\mathrm{d}\tau + f_k(t), \quad t \geq 0, k \in \mathbb{Z}. \tag{1.3}$$

A special case of (1.3) is

$$A_{k-j}(t-z) = V_{k-j} A(t-z),$$

which gives a discrete analog of (1.1).

In the present paper, we are interested in the asymptotic speed of propagation for (1.3), the existence of traveling wave solutions for the corresponding time-translation invariant homogeneous system

$$u_k(t) = \int_{-\infty}^t \sum_{j\in\mathbb{Z}} A_{k-j}(t-\tau) g(u_j(\tau))\mathrm{d}\tau, \quad t \in \mathbb{R}, k \in \mathbb{Z}. \tag{1.4}$$

For the convenience of use, we first give some notations:

(i) $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{R}_+ = [0, +\infty)$, $I_N = \{j \mid |j| \leq N, N \in \mathbb{N}_+\}$;
(ii) $u(t, j) = u_j(t)$, $u(t) = u(t, \cdot) = \{u_j(t)\}_{j\in\mathbb{Z}}$, $\operatorname{supp} u(t) = \{j \mid u(t, j) \neq 0\}$ is the support of $u(t, \cdot)$;
(iii) $u(t) \geq v(t)$ if $u_j(t) \geq v_j(t)$ for $j \in \mathbb{Z}$, $u(t) \succ v(t)$ if $u(t) \geq v(t)$ and $u_j(t) > v_j(t)$ for $j \in \operatorname{supp} v(t)$.

We shall assume that the followings hold through this article.

(H_f) For any $j \in \mathbb{Z}$, $f_j \in C(\mathbb{R}_+, \mathbb{R}_+)$; for any given $t \in \mathbb{R}_+$, $\operatorname{supp} f(t)$ is not empty.
(H_A) For any $j \in \mathbb{Z}$, $t \in \mathbb{R}$, $A_j(t) = A_{-j}(t)$ (we call $\{A_j(t)\}$ is isotropic), $A_j(t) \geq 0$; $A_0(t) > 0$, $A_1(t) > 0$ for some interval $t \in [a, b] \subset \mathbb{R}_+$; let $\mathbb{A}(s) := \sum_{j\in\mathbb{Z}} A_j(s)$, and $\mathbb{A} \in L_1(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+)$ and $\int_0^\infty \mathbb{A}(t)\mathrm{d}t = 1$; there exists a $\bar{\lambda} : 0 < \bar{\lambda} \leq \infty$ such that

$$\int_0^\infty \sum_{j\in\mathbb{Z}} A_j(s) \mathrm{e}^{\lambda j} \mathrm{d}s < \infty$$

for $\lambda \in \Lambda = [0, \bar{\lambda})$;
(H_g) $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and there exists some $L > 0$ such that $|g(u) - g(v)| \leq L|u - v|$ for $u, v \in \mathbb{R}_+$; $g(0) = 0$, $g'(0) > 1$; there is a $p > 0$ such that $g(p) = p$, $g(x) > x$ for $0 < x < p$, $p \leq g(x) \leq x$ for $x > p$.

Remark 1.1. We mention here that one could find Λ for some $\{A_j(t)\}$ in (H_A). In fact, if there exists a positive integer N such that $A_j(t) \equiv 0$ for $|j| \geq N$, $t \in \mathbb{R}$, then $\Lambda = [0, \infty)$; if $A_j(t) = \mathrm{e}^{-\bar{\lambda}|j|} q(t)$ for $j \in \mathbb{Z}$, $\bar{\lambda} > 0$ and some $q(t)$ satisfying $\int_0^\infty q(t)\mathrm{d}t < \infty$, then $\Lambda = [0, \bar{\lambda})$.

The organization of this paper is as follows. In Section 2, we prove the existence, uniqueness, positivity and other properties of solutions for (1.3). In Section 3, we establish the existence of the asymptotic speed of propagation c_* by appealing to the methods in [5,13]. In Section 4, we obtain the existence of monotone traveling waves with wave speed $c \geq c_*$ by the method of upper and lower solutions and a limiting argument, and the nonexistence of traveling waves with wave speed $c \in (0, c_*)$. It turns out that the asymptotic speed of propagation coincides with the minimal wave speed.

2. Solutions of (1.3)

A solution of (1.3) on an interval $I \subset \mathbb{R}_+$ is a sequence of functions $\{u_k(t)\}_{k\in\mathbb{Z}}$ defined on I satisfying (1.3). $u(t)$ is isotropic on I if $u_k(t) = u_{-k}(t)$ for $k \in \mathbb{Z}$, $t \in I$. In the following, we simply call $u(t) = \{u_k(t)\}_{k\in\mathbb{Z}}$, a function $u(t)$.

For any given $\lambda > 0$, define a set

$$S_\lambda = \left\{ u(t) = \{u_k(t)\}_{k\in\mathbb{Z}} \mid u_k \in C(\mathbb{R}_+, \mathbb{R}_+), k \in \mathbb{Z}, \sup_{j\in\mathbb{Z}, t\in\mathbb{R}_+} \{u_j(t)\mathrm{e}^{-\lambda t}\} < \infty \right\}$$

equipped with a norm of $\|u\|_\lambda = \sup_{t\in\mathbb{R}_+, j\in\mathbb{Z}} \{u_j(t)\mathrm{e}^{-\lambda t}\}$. Then S_λ is a Banach space.