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Existence and asymptotic behavior of C^1 solutions to the multi-dimensional compressible Euler equations with damping☆

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Abstract

In this paper, the existence and asymptotic behavior of C^1 solutions to the multi-dimensional compressible Euler equations with damping on the framework of Besov space are considered. Comparing with the well-posedness results of Sideris–Thomases–Wang [T. Sideris, B. Thomases, D.H. Wang, Long time behavior of solutions to the three-dimensional compressible Euler with damping, Comm. Partial Differential Equations 28 (2003) 953–978], we *weaken* the regularity assumptions on the initial data. The global existence lies on a crucial a-priori estimate which is obtained by the *spectral localization* method. The main analytic tools are the Littlewood–Paley decomposition and Bony's paraproduct formula.

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Keywords: Euler equations; Damping; Classical solutions; Spectral localization

1. Introduction and main results

In this paper, we study the following Euler equation with damping for a perfect gas flow:

$$\begin{cases} n_t + \nabla\cdot(n\mathbf{u}) = 0 \\ (n\mathbf{u})_t + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u}) + \nabla p(n) = -an\mathbf{u} \end{cases} \tag{1.1}$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}^N,\ N \geqslant 1$, where n and $\mathbf{u} = (u^1, u^2, \cdots, u^N)^{\mathrm{T}}$ (T represents transpose) denote the density, velocity for the gas respectively. $n\mathbf{u}$ stands for the momentum. The pressure p satisfies the γ-law:

$$p = p(n) = An^{\gamma}, \tag{1.2}$$

where the case $\gamma > 1$ corresponds to the isentropic gas and $\gamma = 1$ corresponds to the isothermal gas, A is a positive constant. The positive constant a is the damping coefficient. The system is supplemented with the initial data

$$(n, \mathbf{u})(x, 0) = (n_0, \mathbf{u}_0)(x), \quad x \in \mathbb{R}^N. \tag{1.3}$$

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The system (1.1) describes that the compressible gas flow passes a porous medium and the medium induces a friction force, proportional to the linear momentum in the opposite direction. It is hyperbolic with two characteristic speeds $\lambda = \mathbf{u} \pm \sqrt{p'(n)}$. As a vacuum appears, it fails to be strict hyperbolic. Thus, the system involves three mechanisms: nonlinear convection, lower-order dissipation of damping and the resonance due to vacuum. After NISHIDA's [1,2] pioneer works for (1.1), many contributions have been made on the small smooth solutions and piecewise smooth Riemann solutions away from vacuum, we can cite [3–8] and their references. Among them, for the one-dimensional case [3], the system can be written in the Lagrangian coordinates as follows:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -au, \end{cases} \tag{1.4}$$

where $v = 1/n$ is the specific volume. It was shown that the system (1.4) was time asymptotically equivalent to the porous media equation. In [8], Sideris, Thomases and Wang explained that the damping only presented weak dissipation in three-dimensional space: it could prevent the development of singularities if the initial data was small and smooth, furthermore, they obtained the decay of classical solutions to the constant background state in L^∞ at a rate of $(1+t)^{-(3/2)}$, but singularities was exhibited for large data under some assumption. However, the main open problems for (1.1) with vacuum are still far from well-known. One of them is to study the singular evolution of the vacuum interface. As the first step in this direction, Xu and Yang [9] proved that a local existence theorem on a perturbation of a planar wave solution for (1.1) under the assumption of physical vacuum boundary condition. For the large-time asymptotic behavior for the solutions with vacuum, recently, Huang and Pan et al. [10,11] gave a complete answer to this problem. In fact, they showed that the L^∞ weak entropy solutions with vacuum for the Cauchy problem converged to the Barenblatt's profile of the porous medium equation strongly in L^p.

In this paper, we are concerned with lowering the regularity of the initial data in the generally multi-dimensional space. As in [8], we also consider a perturbation of the constant equilibrium state $(\bar{n}, 0)(\bar{n} > 0)$. First of all, we give a local existence result in Besov space $B^{\sigma}_{2,1}(\mathbb{R}^N)$ $(\sigma = 1 + \frac{N}{2})$ for (1.1)–(1.3) away from vacuum.

Theorem 1.1 ($N \geqslant 1$). *Suppose that* $(n_0 - \bar{n}, \mathbf{u}_0) \in B^{\sigma}_{2,1}(\mathbb{R}^N)$ *with* $n_0 > 0$, *then there exist a time* $T_0 > 0$ *and a unique solution* $(n, \mathbf{u})$ *of the system* (1.1)–(1.3) *such that*

$$(n, \mathbf{u}) \in C^1([0, T_0] \times \mathbb{R}^N) \quad \text{with } n > 0 \text{ for all } t \in [0, T_0]$$

and

$$(n - \bar{n}, \mathbf{u}) \in C([0, T_0], B^{\sigma}_{2,1}(\mathbb{R}^N)) \cap C^1([0, T_0], B^{\sigma-1}_{2,1}(\mathbb{R}^N)).$$

Remark 1.1. The nonlinear pressure term makes computation more fussy in the *spectral localization* estimates due to commutators. To get around this, we introduce a new variable (sound speed) which transforms the nonlinear term into linear and double-linear terms in virtue of the ideas in [8]. In fact, the original system (1.1)–(1.3) can be transformed into a symmetric hyperbolic system (3.1) and (3.2), which are useful to obtain the effective a-priori estimates. Different from the local existence result in [8], Theorem 1.1 follows from Proposition 4.1, Remarks 3.1 and 4.1. The proof of Proposition 4.1 is organized as follows. First, we regularize the initial data of (3.1) and (3.2) and obtain approximative local solutions based on Kato's results. Second, we find a uniform positive time T_0 such that the approximative solution sequence is uniform bounded in $C([0, T_0]; B^{\sigma}_{2,1}(\mathbb{R}^N)) \cap C^1([0, T_0], B^{\sigma-1}_{2,1}(\mathbb{R}^N))$. Finally, we utilize the compactness argument to pass the limit (for detail, see Proposition 4.1).

Under a smallness assumption, we establish the global existence of classical solutions in Besov space $B^{\sigma+\varepsilon}_{2,2}(\mathbb{R}^N)$ $(\sigma = 1 + \frac{N}{2}, \varepsilon > 0)$ for (1.1)–(1.3).

Theorem 1.2 ($N \geqslant 3$). *Suppose that* $(n_0 - \bar{n}, \mathbf{u}_0) \in B^{\sigma+\varepsilon}_{2,2}(\mathbb{R}^N)$. *There exists a positive constant* δ_0 *depending only on* A, γ, a *and* $\bar{n}$ *such that if*

$$\|(n - \bar{n}, \mathbf{u})(\cdot, 0)\|^2_{B^{\sigma+\varepsilon}_{2,2}(\mathbb{R}^N)} + \|(n_t, \mathbf{u}_t)(\cdot, 0)\|^2_{B^{\sigma-1+\varepsilon}_{2,2}(\mathbb{R}^N)} \leqslant \delta_0, \tag{1.5}$$

then there exists a unique global solution $(n, \mathbf{u})$ *of the system* (1.1)–(1.3) *satisfying*

$$(n, \mathbf{u}) \in C^1([0, \infty) \times \mathbb{R}^N)$$