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Fixed points, coincidence points and maximal elements with applications to generalized equilibrium problems and minimax theory

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Abstract

In this paper, using a generalization of the Fan–Browder fixed point theorem, we obtain a new fixed point theorem for multivalued maps in generalized convex spaces from which we derive several coincidence theorems and existence theorems for maximal elements. Applications of these results to generalized equilibrium problems and minimax theory will be given in the last sections of the paper.
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1. Introduction

In 1961, using his own generalization of the classical Knaster–Kuratowski–Mazurkievicz theorem, Ky Fan [7] established an elementary but very basic "geometric" lemma for multivalued maps. In 1968 Browder [4] obtained a fixed point theorem which is the more convenient form of Fan's lemma. Since then this result has been known as the Fan–Browder fixed point theorem, and numerous generalizations of this have appeared in the literature, first in Hausdorff topological vector spaces and, later, in generalized convex spaces. Many of these generalizations have major applications in nonlinear analysis, game theory and abstract economies.

In this paper, using a generalization of the Fan–Browder fixed point theorem due to Yu and Lin [23] we obtain a new fixed point theorem for multivalued maps in generalized convex spaces from which we derive several coincidence theorems and existence theorems for maximal elements. Applications of these results to generalized equilibrium problems and minimax theory will be given in the last sections of the paper.

Let us describe, in short, these applications.

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Let E be a topological vector space, X be a nonempty subset of E and $f : X \times X \to \mathbb{R}$ be a function with $f(x, x) \geq 0$ for all $x \in X$; then the scalar equilibrium problem in the sense of Blum and Oettli [3] is to find $x_0 \in X$ such that $f(x_0, x) \geq 0$ for all $x \in X$. This problem includes fundamental mathematical problems like optimization problems, variational inequalities, fixed point problems, saddle point problems, problems of Nash equilibria, complementary problems [3]. Recently Lin and Du [14] showed that equilibrium problems contain also Ekeland's variational principle as a special case. In the last few years the scalar equilibrium problem was extensively investigated and generalized to vector equilibrium for single-valued or multivalued mappings [1,5,6,12–16] and references therein).

In many of the papers mentioned above one studies some of the following generalized equilibrium problems:

Let X be a G-convex space (particularly, a convex subset of a topological vector space), Z and V be nonempty sets and $F : X \times Z \multimap V$, $C : X \multimap V$ be multivalued maps. Find $x_0 \in X$ such that one of the following situations occurs:

$F(x_0, z) \subseteq C(x_0)$ for all $z \in Z$;

$F(x_0, z) \cap C(x_0) \neq \emptyset$ for all $z \in Z$;

$F(x_0, z) \nsubseteq C(x_0)$ for all $z \in Z$;

$F(x_0, z) \cap C(x_0) = \emptyset$ for all $z \in Z$.

In Section 4 we try a unified approach for all these problems considering a (binary) relation ρ on 2^V and looking for a point $x_0 \in X$ such that $F(x_0, z)\rho C(x_0)$ for all $z \in Z$.

Since Ky Fan [8] and Liu [17] extended the von Neumann–Sion principle obtaining two-function minimax inequalities, many such results involving two of more functions have appeared in the literature. In the last section we obtain a very general minimax inequality of the following type:

$$\inf_{x\in X} h(x, x) \leq \sup_{x\in X} \inf_{z\in Z} f(x, z) + \sup_{z\in Z} \inf_{x\in X} g(x, z),$$

which is, to the best of our knowledge, different to all the minimax inequalities known in the literature.

2. Preliminaries

A multivalued map (or simply, a map) $T : X \multimap Y$ is a function from X into the power set of a set Y. As usual, the set $\{(x, y) \in X \times Y : y \in T(x), x \in X\}$ is called the graph of T and for $A \subset X$ the set $T(A) = \bigcup_{x\in A} T(x)$ is called the image of A under T. The inverse $T^- : Y \multimap X$ is defined by $x \in T^-(y)$ if and only if $y \in Tx$.

Let X and Y be two topological spaces and $T : X \multimap Y$ be a map. T is said to be *upper semicontinuous* (for short, u.s.c) (respectively, *lower semicontinuous* (for short, l.s.c.)) at $x \in X$ if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$), there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in V(x)$; T is said to be u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; T is *closed* if its graph is a closed subset of $X \times Y$.

The following lemma collects known facts about u.s.c. or l.s.c. maps (see for instance [10] for assertions (i), (ii), (iii), [21] for assertion (iv)).

Lemma 1. *Let X and Y be topological spaces and $T : X \multimap Y$ be a map.*

(i) *If Y is regular and T is u.s.c. with closed values, then T is closed.*
(ii) *If Y is a compact space and T is closed, then T is u.s.c.*
(iii) *If X is a compact and T is u.s.c. with compact values, then $T(X)$ is compact.*
(iv) *T is l.s.c. in $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ converging to x, there exists a net $\{y_\alpha\}$ converging to y, with $y_\alpha \in T(x_\alpha)$ for each α.*

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ (see [20]) consists of a topological space X and a nonempty set D such that for each $A \in \langle D\rangle$ with the cardinality $|A| + 1$ there exists a subset $\Gamma(A)$ of X and a continuous function $\Phi_A : \triangle_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\Phi_A(\triangle_J) \subset \Gamma(J)$. Here $\langle D\rangle$ denotes the set of all nonempty finite subsets of D, $\triangle_n$ denotes any n-simplex with vertices $\{e_i\}_{i=0}^n$ and $\triangle_J$ the face of $\triangle_n$ corresponding to J; that is if $A = \{u_0, u_1, \ldots, u_n\}$ and $J = \{u_{i_0}, u_{i_1}, \ldots, u_{i_k}\} \subset A$ then $\triangle_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

The main example of a G-convex space corresponds to the case when $X = D$ is a convex subset of a Hausdorff topological vector space, and for each $A \in \langle X\rangle$, $\Gamma(A)$ is the convex hull of A. For other major examples of G-convex space, see [18,19].

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