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Linear estimate for the number of limit cycles of a perturbed cubic polynomial differential system

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Abstract

Perturbing the cubic polynomial differential systems $\dot{x} = -y(a_1x + a_0)(b_1y + b_0)$, $\dot{y} = x(a_1x + a_0)(b_1y + b_0)$ having a center at the origin inside the class of all polynomial differential systems of degree n, we obtain using the averaging theory of second order that at most $17n + 15$ limit cycles can bifurcate from the periodic orbits of the center.

MSC: 37G15; 37D45

Keywords: Limit cycle; Averaging theory; Polynomial differential system

1. Introduction

In this paper we consider the following planar differential equations:

$$\begin{aligned} x' &= -y(a_1x + a_0)(b_1y + b_0) + \varepsilon p(x, y), \\ y' &= x(a_1x + a_0)(b_1y + b_0) + \varepsilon q(x, y), \end{aligned} \tag{1}$$

where $a_i, b_i \neq 0$ for $i = 1, 2$, ε is small enough, p and q are two arbitrary polynomials of degree n. The systems have essentially a linear center with two straight lines of singular points when $\varepsilon = 0$.

The distribution of the limit cycles is one of the basic problems in the qualitative theory of real plane differential systems. A classical way to obtain limit cycles is perturbing the periodic orbits of a center, for example perturbing the linear center $\dot{x} = -y$, $\dot{y} = x$, inside the class of polynomial differential systems of degree n; i.e. $\dot{x} = -y + \varepsilon p(x, y)$, $\dot{y} = x + \varepsilon q(x, y)$, where $p(x, y)$ and $q(x, y)$ are arbitrary polynomials of degree at most n. In this case it is well known that at most $[(n-1)/2]$ limit cycles can bifurcate from the periodic orbits of the linear center up to first order in ε; see for instance [2]. Here $[\cdot]$ denotes the integer part function.

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An interesting question is how the number of limit cycles which can bifurcate from the periodic orbits of the center changes when we change the linear center for a linear center having some curves of singular points as the center of system (1) for $\varepsilon = 0$. The answer is that we get more limit cycles bifurcating from the periodic orbits of the center when we add these curves of singular points to the linear center.

A good tool for studying the limit cycles which bifurcate from the periodic orbits of these centers is the averaging method; see [1,7,8]. In [5,6] we perturb the quadratic center formed by the linear center and a straight line of singular points, i.e. $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$, inside the class of polynomial differential systems of degree n. We obtain at least n limit cycles bifurcating from the periodic orbits of the center using averaging theory of first order, and at most $2n-1$ limit cycles using averaging theory of second order. For perturbation of degree $n = 4$ this upper bound $2n-1$ is reached. So the number of limit cycles which can bifurcate from the periodic orbits of the center now that the center has a straight line of singular points has increased, and also increases with the order of the averaging theory that we use for studying this number.

Our goal is to show that this is also the case when we study the linear center with two straight lines of singular points, i.e. the number of limit cycles which can bifurcate from the periodic orbits of the center is now larger than the number of limit cycles which can bifurcate from the linear center and the linear center with a straight line of singular points, and also this number increases if we use the averaging theory of second order instead of the averaging theory of first order.

Moreover in [3] using averaging theory of first order the authors studied the number of limit cycles which bifurcate from the cubic polynomial differential systems

$$\begin{aligned} \dot{x} &= -yf(x,y), \\ \dot{y} &= xf(x,y), \end{aligned} \tag{2}$$

where $f(x,y) = 0$ is a conic such that $f(0,0) \neq 0$, when we perturb them inside the class of cubic polynomial differential systems having the origin as a singular point. In [1] this perturbation is studied also using averaging theory of first order but now inside the class of polynomial differential systems of degree n; then the number of limit cycles which can bifurcate has an upper estimate of $3[(n-1)/2]+4$ if $|a_0/a_1| \neq |b_0/b_1|$, and of $2[(n-1)/2]+2$ if $|a_0/a_1| = |b_0/b_1|$, and lower estimates of $3[(n-1)/2]+2$ and $2[(n-1)/2]+1$, respectively.

In this paper we study the number of limit cycles of the polynomial differential system (1) of degree n which bifurcate from the periodic orbits of the same system when $\varepsilon = 0$. This time we use the averaging theory of second order. Of course the estimate obtained is larger than the previous one obtained using the averaging theory of first order.

Theorem 1. *Using the averaging theory of second order, the upper bound for the number of limit cycles of the polynomial differential system* (1) *of degree n bifurcating from the periodic orbits of the center of system* (1) *with* $\varepsilon = 0$ *is* $17n + 15$.

In general the upper bound obtained in Theorem 1 can be far from the best possible upper bound.

This paper is arranged as follows. In Section 2 we recall some fundamental results on averaging theory. In Section 3 we make some preparations for the proof of Theorem 1. In Section 4 we obtain the second averaged function after tedious computations, and finally in Section 5 we give the proof of Theorem 1 using the argument principle.

2. Averaging theory of second order

In this section we summarize the main results on the theory of averaging that we will apply to systems (2). The next theorem provides a first-order approximation for the periodic solutions of a periodic differential system; for a proof see Theorem 2.6.1 of Sanders and Verhulst [7] and Theorem 11.5 and 11.6 of Verhulst [8]. The original theorems are given for a system of differential equations, but since we will use them only for one differential equation we state them in this case.

Theorem 2. *We consider the following two initial value problems:*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \qquad x(0) = x_0, \tag{3}$$

and

$$\dot{y} = \varepsilon f^0(y), \qquad y(0) = x_0, \tag{4}$$