Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 70 (2009) 483–491
Nonlinear Analysis
www.elsevier.com/locate/na

Existence results for an even-order boundary value problem on time scales

İsmail Yaslan

Department of Mathematics, Pamukkale University, 20070 Denizli, Turkey

Received 19 November 2007; accepted 13 December 2007

Abstract

Let $\mathbb{T}$ be a time scale with $t_1, t_2, t_3 \in \mathbb{T}$. We investigate the existence of solutions to the nonlinear even-order three-point boundary value problem

$$(-1)^n y^{\Delta^{2n}}(t) = f(t, y(\sigma(t))), \quad t \in [t_1, t_3] \subset \mathbb{T},$$
$$y^{\Delta^{2i+1}}(t_1) = 0, \qquad \alpha y^{\Delta^{2i}}(\sigma(t_3)) + \beta y^{\Delta^{2i+1}}(\sigma(t_3)) = y^{\Delta^{2i+1}}(t_2), \quad 0 \le i \le n-1,$$

where $n \in \mathbb{N}$, $t_2 \in (t_1, \sigma(t_3))$, $\alpha > 0$ and $\beta > 1$ are given constants.

MSC: 34B18; 39A10

Keywords: Boundary value problems; Cone; Fixed point theorems; Positive solutions; Time scales

1. Introduction

We are interested in the even-order three-point boundary value problem (TPBVP)

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y(\sigma(t))), & t \in [t_1, t_3] \subset \mathbb{T},\ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(t_1) = 0, & \alpha y^{\Delta^{2i}}(\sigma(t_3)) + \beta y^{\Delta^{2i+1}}(\sigma(t_3)) = y^{\Delta^{2i+1}}(t_2), \end{cases} \tag{1.1}$$

for $0 \le i \le n-1$, where $\alpha > 0$ and $\beta > 1$ are given constants. We assume that $f : [t_1, \sigma(t_3)] \times \mathbb{R} \to \mathbb{R}$ is continuous and that $\sigma(t_3)$ is right dense so that $\sigma^j(t_3) = \sigma(t_3)$ for $j \ge 1$. Throughout this paper we suppose that $\mathbb{T}$ is any time scale and $[t_1, t_3]$ is a subset of $\mathbb{T}$ such that $[t_1, t_3] = \{t \in \mathbb{T} : t_1 \le t \le t_3\}$.

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [9]. For the background and results concerning dynamic equations on time scales, see the excellent text by Bohner and Peterson [5] and their edited text [6].

In recent years, many authors have studied second-order three-point boundary value problems for dynamic equations on time scales. We refer the reader to [1,2,7,11,12,14] for some recent results.

E-mail address: iyaslan@pau.edu.tr.

0362-546X/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2007.12.019

Anderson and Avery [3] were interested in the following even-order TPBVP,

$$\begin{cases} (-1)^n x^{(\Delta\nabla)^n}(t) = \lambda h(t) f(x(t)), & t \in [a, c] \subset \mathbb{T},\ n \in \mathbb{N} \\ x^{(\Delta\nabla)^i}(a) = 0, \qquad x^{(\Delta\nabla)^i}(c) = \beta x^{(\Delta\nabla)^i}(b), & 0 \le i \le n-1. \end{cases} \tag{1.2}$$

They have studied the existence of at least one positive solution to TPBVP (1.2) using the functional-type cone expansion–compression fixed point theorem.

Sang and Su [13] are concerned with the existence of at least one nontrivial solution of the following higher-order TPBVP on time scales

$$u^{\Delta^n}(t) + f(t, u(t)) = 0, \quad t \in (0, T),$$
$$u(0) = \alpha u(\eta), \qquad u(T) = \beta u(\eta),$$
$$u^{\Delta^i}(0) = 0 \quad \text{for } i = 1, 2, \ldots, n-2.$$

We have arranged the paper as follows. In Section 2, we construct the Green's function for the TPBVP (1.1). In Section 3, we establish criteria for the existence of at least one solution and of at least one positive solution for the TPBVP (1.1) by using Schauder fixed point theorem and Krassnoselskii's fixed point theorem, respectively. In Section 4, we investigate the existence of multiple positive solutions to the TPBVP (1.1) by using Avery–Henderson fixed point theorem and Legget–Williams fixed point theorem. As an application, to demonstrate our results we also give an example.

2. Preliminaries

Let $G(t, s)$ be Green's function for the boundary value problem

$$-y^{\Delta^2}(t) = 0, \quad t \in [t_1, t_3],$$
$$y^{\Delta}(t_1) = 0, \qquad \alpha y(\sigma(t_3)) + \beta y^{\Delta}(\sigma(t_3)) = y^{\Delta}(t_2).$$

A direct calculation gives

$$G(t, s) = \begin{cases} H(t, s), & t_1 \le s \le t_2, \\ K(t, s), & t_2 < s \le t_3, \end{cases} \tag{2.1}$$

where

$$H(t, s) = \begin{cases} \sigma(t_3) - t + \dfrac{\beta - 1}{\alpha}, & \sigma(s) \le t, \\ \sigma(t_3) - \sigma(s) + \dfrac{\beta - 1}{\alpha}, & t \le s, \end{cases}$$

and

$$K(t, s) = \begin{cases} \sigma(t_3) - t + \dfrac{\beta}{\alpha}, & \sigma(s) \le t, \\ \sigma(t_3) - \sigma(s) + \dfrac{\beta}{\alpha}, & t \le s. \end{cases}$$

To state and prove the main results of this paper, we need the following lemmas.

Lemma 1. *Let $\alpha > 0$, $\beta > 1$. Then the Green's function $G(t, s)$ in (2.1) satisfies the following inequality*

$$G(t, s) \ge \frac{t - t_1}{\sigma(t_3) - t_1} G(\sigma(t_3), s)$$

for $(t, s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

Proof. We proceed sequentially on the branches of the Green's function $G(t, s)$ in (2.1).

(i) For $s \in [t_1, t_2]$ and $\sigma(s) \le t$, we obtain

$$G(t, s) = \sigma(t_3) - t + \frac{\beta - 1}{\alpha}$$