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Primal–dual interior-point algorithms for second-order cone optimization based on kernel functions[☆]

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ABSTRACT

We present primal–dual interior-point algorithms for second-order cone optimization based on a wide variety of kernel functions. This class of kernel functions has been investigated earlier for the case of linear optimization. In this paper we derive the iteration bounds $O(\sqrt{N}\,\log N)\log\frac{N}{\epsilon}$ for large- and $O(\sqrt{N})\log\frac{N}{\varepsilon}$ for small-update methods, respectively. Here N denotes the number of second-order cones in the problem formulation and ε the desired accuracy. These iteration bounds are currently the best known bounds for such methods. Numerical results show that the algorithms are efficient.

1. Introduction

Second-order cone optimization (SOCO) problems are convex optimization problems because their objective is a linear function and their feasible set is the intersection of an affine space with the Cartesian product of a finite number of second-order (also called Lorentz or ice-cream) cones. The second-order cone in $\mathbf{R}^n$ is given by

$$\mathcal{L}^n := \left\{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \;:\; x_1^2 \geq \sum_{i=2}^{n} x_i^2, x_1 \geq 0\right\}, \tag{1}$$

where n is some natural number. A SOCO problem is a problem of the form

$$(P) \qquad \min\left\{c^{\mathrm{T}}x \;:\; Ax = b, x \in \mathcal{K}\right\},$$

and the dual problem of (P) is given by

$$(D) \qquad \max\left\{b^{\mathrm{T}}y \;:\; A^{\mathrm{T}}y + s = c, s \in \mathcal{K}\right\},$$

where $\mathcal{K} \subseteq \mathbf{R}^n$ is the Cartesian product of several second-order cones, i.e.,

$$\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2 \cdots \times \mathcal{K}^N, \tag{2}$$

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with $\mathcal{K}^j = \mathcal{L}^{n_j}$ for each j, and $n = \sum_{j=1}^{N} n_j$. Furthermore, we partition the vectors x, s, c, and the matrix A accordingly as $x = (x^1; x^2; \ldots; x^N)$, $s = (s^1; s^2; \ldots; s^N)$ with $x^j, s^j \in \mathcal{K}^j$, $c = (c^1; c^2; \ldots; c^N)$ with $c^j \in R^{n_j}$, and $A = (A^1, A^2, \ldots, A^N)$ with $A^j \in \mathbf{R}^{m\times n_j}$, and $b \in \mathbf{R}^m$. Without loss of generality, throughout the paper we assume that the matrix A has full rank, (i.e., rank $(A) = m$). As a consequence, if the pair (y, s) is dual feasible then y is uniquely determined by s. Therefore, we will feel free to say that s is dual feasible, without mentioning y.

It is well-known that SOCO problems include linear and convex quadratic programs as special cases. On the other hand, SOCO problems are special cases of semidefinite optimization (SDO) problems, and hence can be solved by using an algorithm for SDO problems. Interior-point methods (IPMs) that exploit the special structure of SOCO problems, however, have much better complexity than when using an IPM for SDO for solving SOCO problems.

In the last few years the SOCO problems have received considerable attention from researchers because of its wide range of applications (see, e.g., [10,27]) and because of the existence of efficient IPM algorithms (see, e.g., [2,3,23,22,24,25]). Several IPMs designed for LO (see e.g., [20]) have been successfully extended to SOCO. Important work in this direction was done by Nesterov and Todd [13,14] who showed that the primal–dual algorithm maintains its theoretical efficiency when the nonnegativity constraints in LO are replaced by a convex cone, as long as the cone is homogeneous and self-dual. Adler and Alizadeh [1] studied a unified primal–dual approach for SDO and SOCO, and proposed a search direction for SOCO analogous to the AHO-direction for SDO. Later, Schmieta and Alizadeh [21] presented a way to transfer the Jordan algebra associated with the second-order cone into the so-called Clifford algebra in the cone of matrices and then carried out a unified analysis of the analysis for many IPMs in symmetric cones. Faybusovich [7], using Jordan algebraic techniques, analyzed the Nesterov–Todd method for SOCO. Monteiro [12] and Tsuchiya [25] applied Jordan algebra to the analysis of IPMs for SOCO with specialization to various search directions. Other researchers have worked on IPMs for special cases of SOCO, such as convex quadratic programming, minimizing a sum of norms, etc. For an overview of these results we refer to [27] and its related references.

Recently, Peng et al. [18] designed primal–dual interior-point algorithms for LO, SDO and SOCO based on so-called self-regular (SR) proximity functions. They improved the iteration bound for SOCO with large-update methods and achieved the currently best bound for such methods. Their work was extended to other proximity functions based on univariate so-called kernel functions. This was done for a wide class of kernel functions in [4] for LO and for one specific kernel in [26] for SDO.

Motivated by these results, in this paper we present a primal–dual IPM for SOCO problems based on kernel functions. We call a univariate $\psi : (0, \infty) \to [0, \infty)$ a kernel function if it satisfies

$$\psi'(1) = \psi(1) = 0, \tag{3a}$$

$$\psi''(t) > 0, \tag{3b}$$

$$\lim_{t\to 0} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty. \tag{3c}$$

Note that this implies that $\psi(t)$ is strictly convex and minimal at $t = 1$, with $\psi(1) = 0$. Moreover, (3c) expresses that $\psi(t)$ has the barrier property. Also note that $\psi(t)$ is completely determined by its second derivative, because the above properties imply that

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}\xi. \tag{4}$$

Similarly as in [4] for LO, we show in this paper that every kernel function $\psi(t)$ gives rise to an IPM for SOCO. We borrow several tools for the analysis of the resulting IPM for SOCO from [4], and some of them from [18]. These analytic tools reveal that the iteration bound highly depends on the choice of $\psi(t)$, especially on the inverse functions of $\psi(t)$ and its derivatives. Our aim will be to investigate the dependence of the iteration bound on the underlying kernel function. We will consider both large- and small-update methods.

The outline of the paper is as follows. In Section 2, after briefly recalling some relevant properties of the second-order cone and its associated Euclidean Jordan algebra, we review some basic concepts for IPMs for solving the SOCO problem, such as central path, NT-search direction, etc. We conclude this section by presenting a generic primal–dual IPM for SOCO based on a kernel function. In Section 3 we introduce the notion of an *eligible* kernel function and review its relevant properties. We define the related vector-valued barrier function and the corresponding real-valued barrier function. Then, in Section 3.2, we derive a crucial inequality, related to the decrease of the barrier function value during an *inner* iteration of the algorithm. At this stage the analysis in essence boils down to the same analysis that was used in [4] for IPMS for LO. Borrowing many results from [4] we complete the analysis of the IPM for SOCO and obtain a generic iteration bound that is completely expressed in terms of the underlying kernel function and two parameters of the algorithm. We conclude this section by applying the iteration bound to a wide variety of kernel functions. Numerical results are described in Section 4. Finally, some concluding remarks follow in Section 5.

Some notations used throughout the paper are as follows. $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of all vectors (with n components), the set of nonnegative vectors and the set of positive vectors, respectively. As usual, $\|\cdot\|$ denotes the Frobenius norm for matrices, and the 2-norm for vectors; this norm is never understood in the algebraic sense (as a square root of sum of squared eigenvalues). The Löwner partial ordering "$\succeq_{\mathcal{K}}$" of $\mathbf{R}^n$ defined by a cone $\mathcal{K}$ is defined by $x \succeq_{\mathcal{K}} s$ if $x - s \in \mathcal{K}$. The interior of $\mathcal{K}$ is denoted as $\mathcal{K}_+$ and we write $x \succ_{\mathcal{K}} s$ if $x - s \in \mathcal{K}_+$. Finally, $\mathbf{E}_n$ denotes the $n \times n$ identity matrix.