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Existence of multiple solutions for weighted $p(r)$-Laplacian equation Dirichlet problems[☆]

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ABSTRACT

This paper deals with the existence of multiple solutions for weighted $p(r)$-Laplacian equation Dirichlet problems. Several sub-super-solution type results have been given via Leray–Schauder's degree method. In particular, we give a counterexample which means that Ω_{α_2} in paper [J. Henderson, H.B. Thompson, Existence of multiple solutions for second order boundary value problems, J. Differential Equations 166 (2000) 443–454] is not an open subset of C_0^1.

1. Introduction

In this paper, we consider the existence of solutions for the following weighted $p(r)$-Laplacian ordinary equation Dirichlet problem via Leray–Schauder's degree method

$$-(w(r)\left|u'\right|^{p(r)-2}u')'+f\left(r,u,(w(r))^{\frac{1}{p(r)-1}}u'\right)=0,\quad r\in(T_1,T_2),\tag{1}$$

$$u(T_1)=0=u(T_2),\tag{2}$$

where $p,w\in C([T_1,T_2],\mathbb{R})$, $p(r)>1$, $0<w(r)$ for any $r\in(T_1,T_2)$ and $[w(r)]^{\frac{-1}{p(r)-1}}\in L^1(T_1,T_2)$, $-\triangle_{p(r)}:=-(w(r)\left|u'\right|^{p(r)-2}u')'$ is called weighted $p(r)$-Laplacian.

The study of differential equations and variational problems with nonstandard $p(r)$-growth conditions is a new and interesting topic. For the background about these problems, we refer to [3,17,23]. Many results have been obtained on this kinds of problem, for example [6–9,11,13,15,17–23]. If $p(r)\equiv p$ (a constant), (1) is the well known p-Laplacian problem. There are many sub-super-solution type results on the existence of solutions for p-Laplacian problems (see [1,2,4,12,16]). Because of the nonhomogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of p-Laplacian, many methods and results for p-Laplacian problems are invalid for $p(x)$-Laplacian problems; for example:

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(i) If $\Omega \subset \mathbb{R}^n$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [9]), but the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

(ii) If $w(r) \equiv p(r) \equiv p$ (a constant) and $-\triangle_p u > 0$, then u is concave, this property is used extensively in the study of one dimensional p-Laplacian problems, but it is invalid for $-\triangle_{p(r)}$. This is another difference on $-\triangle_p$ and $-\triangle_{p(r)}$.

Regarding the existence of multiple solutions for $p(x)$-Laplacian equations via critical point theory, we refer to [8]. For the existence of solutions for $p(x)$-Laplacian equations Dirichlet problems via Leray–Schauder degree method or sub-super-solution method, we refer to [20–22]. This paper deals with the existence of multiple solutions for weighted $p(r)$-Laplacian equation Dirichlet problems, several sub-super-solution type results have been given via Leray–Schauder degree method. This paper is motivated by [10,14,20].

Let $T_1 < T_2$ and $I = [T_1, T_2]$, the function $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Caratheodory, by this we mean:

(i) for almost every $t \in I$ the function $f(t, \cdot, \cdot)$ is continuous;
(ii) for each $(x, y) \in \mathbb{R} \times \mathbb{R}$ the function $f(\cdot, x, y)$ is measurable on I;
(iii) for each $R > 0$ there exists a $\rho_R \in L^1(I, \mathbb{R})$ such that, for almost every $t \in I$ and every $(x, y) \in \mathbb{R} \times \mathbb{R}$ with $|x| \le R$, $|y| \le R$, one has

$$|f(t, x, y)| \le \rho_R(t).$$

We set $C = C(I, \mathbb{R})$, $C^1 = \{u \in C \mid u' \in C((T_1, T_2), \mathbb{R}), \lim_{r \to T_1^+} w(r) \left|u'\right|^{p(r)-2} u'(r)$ and $\lim_{r \to T_2^-} w(r) \left|u'\right|^{p(r)-2} u'(r)$ exist$\}$, $C_0^1 = \{u \in C^1 \mid u(T_1) = 0 = u(T_2)\}$. Denote $\| u \|_0 = \sup_{r \in (T_1, T_2)} |u(r)|$ and $\| u \|_1 = \| u \|_0 + \| (w(r))^{\frac{1}{p(r)-1}} u' \|_0$. The spaces C and C^1 will be equipped with the norm $\| \cdot \|_0$ and $\| \cdot \|_1$, respectively. Then C, C^1 and C_0^1 are Banach spaces.

Throughout the paper, we denote

$$w(T_1) \left|u'\right|^{p(T_1)-2} u'(T_1) = \lim_{r \to T_1^+} w(r) \left|u'\right|^{p(r)-2} u'(r),$$

$$w(T_2) \left|u'\right|^{p(T_2)-2} u'(T_2) = \lim_{r \to T_2^-} w(r) \left|u'\right|^{p(r)-2} u'(r).$$

We say a function $u : I \to \mathbb{R}$ is a solution of (1) with (2), if $u \in C_0^1$ and $w(r) \left|u'\right|^{p(r)-2} u'(r)$ is absolutely continuous and satisfies (1) *a.e.* on I.

Functions $\alpha, \beta \in C^1$ are called subsolution and supersolution of (1) respectively, if $w(r) \left|\alpha'\right|^{p(r)-2} \alpha'(r)$ and $w(r) \left|\beta'\right|^{p(r)-2} \beta'(r)$ are absolutely continuous and satisfy

$$-(w(r) \left|\alpha'\right|^{p(r)-2} \alpha'(r))' + f(r, \alpha, (w(r))^{\frac{1}{p(r)-1}} \alpha') \le 0, \quad \text{a.e. on } I,$$

$$-(w(r) \left|\beta'\right|^{p(r)-2} \beta'(r))' + f(r, \beta, (w(r))^{\frac{1}{p(r)-1}} \beta') \ge 0, \quad \text{a.e. on } I.$$

Throughout the paper, we assume that $\alpha_1 \le \beta_2$ are subsolution and supersolution, respectively. Denote

$$\Omega_0 = \{(t, x) \mid t \in I, x \in [\alpha_1(t), \beta_2(t)]\},$$

$$\Omega_1 = \{(t, x, y) \mid t \in I, x \in [\alpha_1(t), \beta_2(t)], y \in \mathbb{R}\}.$$

We also assume f satisfies the following conditions on Ω_0 and Ω_1,

(H_1) $|f(t, x, y)| \le A_1(t, x)K_1(t, x, y) + A_2(t, x)K_2(t, x, y)$ for all $(t, x, y) \in \Omega_1$, where $A_i(t, x)$ $(i = 1, 2)$ are positive valued and continuous on Ω_0, $K_i(t, x, y)$ $(i = 1, 2)$ are positive valued and continuous on Ω_1.

(H_2) There exist positive numbers M_1 and M_2 such that

$$K_1(t, x, y) \le |y| \, h(|y|), K_2(t, x, y) \le M_1 h(|y|), \quad \forall \, |y| \ge M_2,$$

where $h \in C([1, +\infty), [1, +\infty))$ is increasing and satisfies $\int_1^{+\infty} \frac{1}{h((y)^{\frac{1}{p^- - 1}})} dy = \infty$, where $p^- = \min_{r \in I} p(r)$.

Our main results are as following

Theorem 1.1. *If f is Caratheodory and satisfies* (H_1) *and* (H_2) *on Ω_0 and Ω_1, α_1 and α_2 are subsolutions, β_1 and β_2 are supersolutions, which satisfy*

(i) $\alpha_i(T_j) \le 0 \le \beta_i(T_j)$, $i = 1, 2$, $j = 1, 2$,