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The topological nature of Krasnoselskii's cone fixed point theorem

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Abstract

In recent years, the Krasnoselskii fixed point theorem for cone maps and its many generalizations have been successfully applied to establish the existence of multiple solutions in the study of boundary value problems of various types. In this article we discuss the topological nature of the Krasnoselskii theorem and show that it can be restated in a more general form without reference to a cone structure or the norm of the underlying Banach space. This new perspective brings out a closer relation between the Krasnoselskii Theorem and the classical Brouwer fixed point theorem. It also points to some obvious extensions of the cone theorem.

Keywords: Krasnoselskii; Cone fixed point theorem

1. Introduction

The classical Brouwer–Schauder fixed point theorem is an undeniably important tool in the study of the existence of solutions to mathematical problems (see e.g. [4,6,14]). In recent years, another fixed point theorem due to Krasnoselskii [7,8] and its generalizations have been successfully applied to obtain existence results for multiple positive solutions of various types of boundary value problems, notably in the case of ordinary differential equations and their discrete versions. Krasnoselskii himself [8] has applied his result to study the existence of periodic solutions of periodic systems of ordinary differential equations.

The main impetus for seeking new cone fixed point theorems is to apply them to obtain better criteria for the existence of solutions, for whatever problems the authors are currently interested in. Krasnoselskii's theorem has two parts (to be described in Section 2). The first part, called the compressive form, bears resemblance to the Brouwer–Schauder theorem. In fact, in [9], we show that the former is a special case of a generalized Brouwer–Schauder theorem. The second part, the expansive form, complements the compressive form. A proof of the expansive form from the compressive form has also been found, but it has not yet been published.

I believe that one of the reasons why the close relationship between the Krasnoselskii's theorem and the Brouwer–Schauder theorem has been overlooked is that the former is usually stated in the setting of a cone embedded in a Banach space with a given norm. In this setting, the norm functional plays a couple of important roles: in defining the region of points we are interested in, and in stating the properties of the images under the given map. When

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attempting to extend Krasnoselskii's theorem, one naturally focuses on finding similar functionals to replace the norm while still preserving these roles. On the other hand, the Brouwer–Schauder theorem is more topological in nature, being free from the concept of a metric. One can easily be misguided by this fact to think that the Brouwer–Schauder theorem is not adequate to deal with the metric aspects of cone maps.

The goal of this paper is to point out that Krasnoselskii's theorem can indeed be interpreted in a non-metric framework. The norm function is more of a convenience rather than a necessity. There are simpler ways to generalize the theorem without using functionals.

In Section 2, we state a simplified version of Krasnoselskii's theorem and discuss several generalizations, especially the Krasnoselskii–Benjamin theorem. In Section 3, we discuss the topological nature of the simplified Krasnoselskii theorem and show that it is equivalent to a fixed point theorem for cylinder maps. This perspective makes it clear how we can formulate a generalized expansive cone result, which incidentally reads more like a Brouwer-type theorem than a cone theorem.

Proofs of the results stated in this paper are omitted, but they can be found in the preprint [10].

2. Krasnoselskii's theorem

The excellent expository article by Amann [1] (Chapter 11) has a discussion and proof of the Krasnoselskii theorem, with the general boundary conditions (2.7) and (2.8). See also [5,10].

Let X be a (finite- or infinite-dimensional) Banach space with a given norm $\|.\|$, and $K \subset X$ be a closed convex cone defined in the usual way, namely, that K satisfies the following conditions.

(K1) If $x \in K$, then $\lambda x \in K$ for all real numbers $\lambda > 0$.
(K2) If $x, y \in K$, then $x + y \in K$.
(K3) If both x and $-x \in K$, then $x = 0$.
(K4) K is closed.

For visualization, we can use the special case where X is the three-dimensional space R^3 with the Euclidean norm, and K is an infinite circular cone with its vertex at the origin, or, even more simply, use the case where X is the two-dimensional plane R^2 and K is the wedge-shaped region AOB in Fig. 1 or 2.

A cone map on K is a completely continuous map $T : K \to K$ (of K into itself). When X is finite-dimensional, any continuous map is completely continuous. A point $x \in K$ is a fixed point of T if $T(x) = x$.

Let $0 < a < b$ be two given numbers. We are interested in conditions which guarantee that T has a fixed point in the annular region $K(a, b) = \{x \in K : a \le \|x\| \le b\}$. Note that $K(a, b)$ is in general not convex, even though K is. We denote by $K_a = \{x \in K : \|x\| = a\}$ and $K_b = \{x \in K : \|x\| = b\}$ the inner and outer boundaries, respectively, of $K(a, b)$. We can extend the notation to define $K(0, a)$ and $K(b, \infty)$ in the obvious way. Theorem 1 is a simplified version of Krasnoselskii's original theorem. An illustration of this result in dimension 2 is depicted in Figs. 1 and 2.

Theorem 1 (*Krasnoselskii 1960 [7]*). *Let $K(a, b)$, T, K_a, and K_b be as defined above.*

1. *(Compressive Form) T has a fixed point in $K(a, b)$ if*

$$\|T(x)\| \ge \|x\| \quad \textit{for all } x \in K_a, \tag{2.1}$$

and

$$\|T(x)\| \le \|x\| \quad \textit{for all } x \in K_b. \tag{2.2}$$

2. *(Expansive Form) T has a fixed point in $K(a, b)$ if*

$$\|T(x)\| \le \|x\| \quad \textit{for all } x \in K_a, \tag{2.3}$$

and

$$\|T(x)\| \ge \|x\| \quad \textit{for all } x \in K_b. \tag{2.4}$$

Note that conditions (2.1)–(2.4) are imposed only on points on the two curved boundaries of $K(a, b)$. Interior points and points on the sides of the cone can be moved in any direction (as long as the image remains inside K). Also it is not stipulated that any particular image point $T(x)$ must lie inside $K(a, b)$.