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Regularity of higher energies of wave equation with nonlinear localized damping and a nonlinear source

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Dedicated to Prof. Lakshmikantham on the occasion of his 85th birthday

Abstract

Wave equation driven by a nonlinear dissipative source and subjected to a nonlinear damping that is localized on a small region near the boundary is considered. While finite-energy ($H^1 \times L_2$) solutions to this problem are bounded uniformly for all times $t > 0$, this property generally fails for higher energy norms ($H^2 \times H^1$). This is due to a "generation" of higher energy by the source and the ultimate loss of dissipativity. However, the presence of the damping may turn things around by counteracting the sources also at the higher energy levels. The benefits of this counteraction depend on the nonlinear characteristics of dissipation. Any deviation from linearity (be it origin or infinity) causes degradation of the damping, hence of decay rates of finite energy. This, in turn, is shown to have an adverse effect on the stability of higher energies. The main aim of this paper is to provide a quantitative analysis of the interaction between nonlinearity of the damping and nonlinearity of the source. We show that under some correlation between growth rates of the damping and the source, the norms of topological order *above the finite-energy level* remain globally bounded for all times.

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1. Introduction

Stability and energy decay rates for dissipative wave equations, as well as other hyperbolic-like structures (plates, shells), have attracted considerable attention in the past years, see books [5,6,8,7,9] and the references therein. The problem of interest in this paper is a semilinear wave equation driven by a nonlinear source $f \in C^1(\mathbb{R})$ and monotone nonlinear dissipation $g \in C(\mathbb{R})$, $g(0) = 0$:

$$w_{tt}(x,t) - \Delta w(x,t) + \chi(x) g(w_t(x,t)) = f(w(x,t)), \quad \{x,t\} \in \Omega \times [0,T[\tag{1}$$

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on a smooth domain $\Omega \subset \mathbb{R}^n$. The growth of the source feedback map $f(s)$ can reach the critical Sobolev's exponent, while the dissipation $\chi g(w_t)$ is geometrically restricted to the support of the cutoff function $\chi(x)$. Eq. (1) will be equipped with appropriate homogeneous boundary conditions specified later.

It is known that under certain dissipativity properties imposed on the source, *finite-energy* $(H^1(\Omega) \times L_2(\Omega))$ solutions decay asymptotically to zero. When the dissipation $\chi g(t)$ satisfies suitable geometric and topological conditions, the decay rates are uniform in the topology of the phase space (finite-energy space). The topological and geometric conditions needed are the following: (i) the support of χ must be sufficiently large, (ii) the function g must obey a linear type of the growth at infinity. Under these conditions the rate of decay of finite energy is completely determined by the behavior of the dissipation at the *origin*. Any deviation from linearity (be it sublinear or superlinear) deteriorates (from exponential) decay properties of finite-energy solutions [10,12]. Deviation from linearity at *infinity* not only "slows down" the decay rates but also requires *higher – above finite energy – regularity of solutions* (thus, the obtained decay rates are no longer uniform with respect to the phase space). It is the lack of equipartition between the potential and kinetic energy components that destroys uniformity (in the same finite-energy topology) of the decay rates: either we have too much of kinetic damping (superlinear case) or too little (sublinear case). In this situation, asymptotic behavior of the finite energy strongly depends on the topological properties of the flow; to claim decay rates at the phase-space level, one must establish *global-in-time* bounds on solutions in topology strictly *above* the order of the finite energy itself. These higher-order norms are what we shall henceforth refer to as the "*higher energy*" which can be associated with $H^2(\Omega) \times H^1(\Omega)$ topology. Paper [12] presents a detailed account on how additional regularity of "high energy" solutions reflects on stability when the dissipative feedback behaves sublinearly or superlinearly at infinity.

This motivates the following question that we ask in this paper:

When are higher-order norms globally bounded in time? With the estimate uniform with respect to higher norms of the initial data.

We know that given sufficiently smooth initial data, higher energy of solutions remains bounded on every finite time interval (e.g. see the appendix in [13]). However, it is far from being clear whether the bound remains uniform for all times. This problem, classical within the realm of dynamic systems, has a simple solution when the system is contractive (no source present in the model). Indeed, nonlinear semigroup theory [2] provides an affirmative answer. However, in the non-contractive case there is no natural mechanism to ensure stability of higher energies, since *dissipativity of the source is guaranteed only at the level of finite-energy space*.

At the level of higher-order norms the nonlinear source term actively "pumps" energy into the system. In general, the best one can claim is that the higher energies obey an exponential bound that blows up as $t \to \infty$. Thus, even though it is relatively easy to prove that the finite energy is globally bounded (and uniformly with respect to norm of the initial state), the norms in finer topologies do not need to obey any global estimates. A heuristic argument indicates that *higher level regularity* depends on the strength of decay rates at the *finite-energy level*. Since the said decay rates depend on the damping and they degraded with damping's deviation from linearity — the problem we address is that of quantitative analysis of the relation between the nonlinearity of the damping and stability of higher energies. In other words the question we ask is the following: *given "strength" of the source, how much deviation from linearity of the damping can be tolerated to guarantee the infinite life span of higher energy solutions?*

The expected answer must be conditional and should depend on the interaction between the source and the damping, in particular it relies on

1. *how strong the source is at the higher energy level* and 2. *how fast the solutions decay to zero at the lower ("finite") energy level.* It is the balance between these two factors that provides an answer to the question asked. We set up an optimization problem between rapid decays and high regularity needed to compensate for the effect of the source. When expressed quantitatively it naturally leads to a fixed point-type argument that involves optimization of regularity and decay estimates. In this paper we show how to resolve this optimization problem and state what conditions can be imposed on the interaction between the source and damping, in order to affirmatively answer the question asked (see Theorem 2).

Remark 1. The methods presented in this paper can be easily adapted to other dissipative problems such as wave equation with boundary damping or Schrodinger and plate equation with either localized or boundary damping.