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Fixed-energy inverse scattering

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Abstract

The author's method for solving inverse scattering problem with fixed-energy data is described. Its comparison with the method based on the D-N map is given. A new inversion procedure is formulated.

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1. Introduction

The inverse scattering problem (ISP) has a long history. Fundamental results for 1D inverse spectral and scattering problems were obtained by Gel'fand–Levitan, Krein and Marchenko. These results are presented in the original monographs [3,4] and in [28], Chapter 3, where many novel results can be found, and the first presentation of Krein's theory with detailed proofs is given. The 3D ISP with fixed-energy scattering data has been open for several decades, from approximately 1943, when the question about the possibility to identify the Hamiltonian from the S-matrix was posed. The ISP consists of finding a potential $q = q(x)$ from the knowledge of the corresponding scattering amplitude $A(\alpha', \alpha)$, known at a fixed energy $k^2 > 0$ for all $\alpha', \alpha \in S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$. One can state ISP in $\mathbb{R}^n$ with $n \geq 2$, but we discuss here only the case $n = 3$. The results are basically the same for other n. Due to lack of space we discuss the inverse potential scattering and refer to [28,29] for inverse obstacle scattering. Let us define the scattering amplitude. Assume that $q \in Q := Q_a \cap L^\infty(\mathbb{R}^3)$, where $Q_a := \{q : q(x) = \overline{q(x)}, q(x) \in L^2(B_a), q(x) = 0 \text{ if } |x| \geq a\}$, $B_a := \{x : |x| \leq a\}$, $a > 0$, and the overbar stands for complex conjugate. The scattering solution is the unique solution of the scattering problem:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in } \mathbb{R}^3, k = \text{const} > 0, u = u_0 + v, \tag{1.1}$$

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where $u_0 := e^{ik\alpha\cdot x}$ is the incident plane wave, $\alpha \in S^2$ is given, and v satisfies the radiation condition: $v_r - ikv = o(r^{-1})$ as $r := |x| \to \infty$. If $q \in Q$, this implies the following asymptotic relation:

$$v = A(\alpha', \alpha)\frac{e^{ikr}}{r} + o(r^{-1}), \quad r := |x| \to \infty, \alpha' := \frac{x}{r}. \tag{1.2}$$

The coefficient $A(\alpha', \alpha) := A_q(\alpha', \alpha)$ is called the scattering amplitude. We drop its k-dependence of $A(\alpha', \alpha) := A(\alpha', \alpha, k)$ since $k > 0$ is fixed. The ISP consists of finding q from the corresponding scattering amplitude $A(\alpha', \alpha)$, known for all $\alpha', \alpha \in S^2$. The basic questions are:

(a) Does this knowledge determine q uniquely?

(b) If yes, how does one calculate q given $A(\alpha', \alpha)$ for all $(\alpha', \alpha) \in S^2$?

(c) If $A_\delta(\alpha', \alpha)$, the "noisy data", are given, such that

$$\sup_{\alpha', \alpha \in S^2} |A_\delta(\alpha', \alpha) - A(\alpha', \alpha)| < \delta,$$

how does one calculate a stable approximation to q, i.e., a q_δ, such that

$$\|q - q_\delta\| < \eta(\delta),$$

where $\eta(\delta) \to 0$ as $\delta \to 0$ and $\|\cdot\|$ is some norm? How can one estimate the rate at which $\eta(\delta)$ tends to 0?

These three questions are: uniqueness, inversion methods for exact and noisy data, and stability of the inversion methods.

In 1987 the author proved a uniqueness theorem for 3D ISP with fixed-energy data corresponding to $q \in Q_a$ and gave an inversion method for exact data [11,13]. In [18,19] an inversion method for noisy data was developed and its error estimate was obtained for $q \in Q$. The key notion the author has introduced for the study of uniqueness of the solution to ISP and to many other inverse problems was the notion of Property C for a pair of differential operators, [10–12,14–17,20]. A summary of these results was presented in [17,25,28]. In [6,1,7,8] a method, Newton–Sabatier (NS) method, for finding a spherically-symmetric potential from the knowledge of the corresponding fixed-energy phase shifts was proposed. In [26] a detailed analysis of this method was given and it was proved that the NS method is fundamentally wrong in the sense that its foundations are wrong (see also [24]). In [2] a uniqueness result is claimed for a version of the NS method. In [27] a counterexample to this result is given (see also [23]). An additional discussion of the NS method can be found in [21]. In Section 2 Ramm's inversion method is described. In Section 3 an inversion method which uses the Dirichlet-to-Neumann (DN) map is described and compared with the author's method. The inversion method, based on the usage of DN map, was proposed in [9], see also [5,12,22,28].

2. Ramm's inversion method for exact data

The results in this Section are taken from [28,19,25]. Let $q \in Q$ and $A(\alpha', \alpha)$ be the scattering amplitude at a fixed energy $k^2 > 0$. In what follows, we take $k = 1$ without loss of generality. One has:

$$A(\alpha', \alpha) = \sum_{\ell=0}^{\infty} A_\ell(\alpha) Y_\ell(\alpha'), \qquad A_\ell(\alpha) := \int_{S^2} A(\alpha', \alpha)\overline{Y_\ell(\alpha')}\mathrm{d}\alpha', \tag{2.1}$$

where S^2 is the unit sphere in $\mathbb{R}^3$, $Y_\ell(\alpha') = Y_{\ell,m}(\alpha'), -\ell \le m \le \ell$, are the normalized spherical harmonics, summation over m is understood in (2.1) and in (2.8) below. Define the following algebraic variety in $\mathbb{C}^3$:

$$M := \{\theta : \theta \in \mathbb{C}^3, \theta \cdot \theta = 1\}, \qquad \theta \cdot w := \sum_{j=1}^{3} \theta_j w_j. \tag{2.2}$$

This variety is non-compact, intersects $\mathbb{R}^3$ over S^2, and, given any $\xi \in \mathbb{R}^3$, there exist infinitely many $\theta, \theta' \in M$ such that

$$\theta' - \theta = \xi, \quad |\theta| \to \infty, \theta, \theta' \in M. \tag{2.3}$$