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# Fixed point theorems for mixed monotone operators with PPF dependence

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**Abstract**

In this paper, fixed point theorems for mixed monotone operators are investigated by weakening the requirements in the contractive assumption and strengthening the metric space utilized with a partial order, taking the domain space $C[[a, b], E]$ different from the range $E$ and using the method of reflection operators.


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## 1. Introduction

The contraction mapping principle and the monotone iterative technique are well known and are independently applicable to a variety of problems. In recent years, there has been a trend to unify the two methods in order to obtain more flexibility in applications. This is done by weakening the requirements in the contractive assumption and strengthening the metric space utilized with a partial order [5,6]. The fixed point results that are generated by this unification are then applied to find existence results for periodic boundary value problems (PBVP) [5].

The contraction mapping theorem was extended in [1] where the domain of the nonlinear operator involved is $C[[a, b], E]$, and the range is $E$, $E$ being a metric space or a Banach space. This result is known as the contraction theorem for operators with PPF dependence [1]. Recently, this result was generalized in the framework of unifying the two methods mentioned above for nondecreasing operators with PPF dependence and then applied to PBVP for delay differential equations [2].

In this paper, we further extend the results of [2] to mixed monotone operators, generalizing fixed point theorems as in [3], and include an application to the PBVP for delay differential equations. Instead of employing a direct proof as in [3], we utilize the trick of using the reflection operator [4] and the expansion of the spaces involved to convert the present problem to a problem for which results are already available in [2], thereby reducing the proofs considerably.

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## 2. Fixed point theorems for mixed monotone operators with PPF dependence

We consider a partially ordered metric space $(E, d)$. Let $E_0 = C[[a, b], E]$ with the corresponding metric $d_0(\phi, \psi) = \max_{a \le s \le b} d[\phi(s), \psi(s)]$. Let $T$ be an operator from $E_0$ to $E$. Then by a fixed point of $T$ we mean there exists a $\phi \in E_0$ such that $T\phi = \phi(c)$ for some $c \in [a, b]$. In order to prove the existence of a fixed point for $T$ we utilize a mixed monotone operator.

Let $F : E_0 \times E_0 \to E$ be such that $F(\phi, \phi) = T\phi, \phi \in E_0$.

Suppose for $\phi_1, \phi_2 \in E_0$, $F(\phi_1, \psi) \le F(\phi_2, \psi)$ whenever $\phi_1 \le \phi_2$, and for $\psi_1, \psi_2 \in E_0$, $F(\phi, \psi_1) \ge F(\phi, \psi_2)$ whenever $\psi_1 \le \psi_2$, then $F$ is known as a mixed monotone operator.

If there exist $\phi^*, \psi^* \in E_0$ satisfying

$$F(\phi^*, \psi^*) = \phi^*(c) \quad \text{and} \quad F(\psi^*, \phi^*) = \psi^*(c),$$

then we say that $\begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix}$ is a coupled fixed point of $F$.

Instead of using a direct proof as in [3], we shall employ the notion of a reflection operator [4], and the expansion of the given space to convert the present problem to a problem to which the known result for fixed point theorems of monotone nondecreasing operators with PPF dependence given in [2] can be applied.

**Theorem 2.1.** *Suppose $F : E_0 \times E_0 \to E$. Assume that*

(i) *$F$ is continuous;*
(ii) *$F$ satisfies the mixed monotone property;*
(iii) *$d[F(\phi, \psi), F(\psi, \phi)] \le k d_0(\phi, \psi)$ for $0 \le k < 1$;*
(iv) *$\alpha_0, \beta_0$ are coupled lower and upper solutions of $F$, namely,*

$$\alpha_0(c) \le F(\alpha_0, \beta_0) \quad \text{and}$$
$$\beta_0(c) \ge F(\beta_0, \alpha_0).$$

*Then, there exist $\phi^*, \psi^* \in E_0$ such that $\phi^*(c) = F(\phi^*, \psi^*)$ and $\psi^*(c) = F(\psi^*, \phi^*)$.*

**Proof.** Before we proceed with the proof, we need the reflection operator $P : E_0 \times E_0 \longrightarrow E_0 \times E_0$ where

$$P(\phi, \psi) = (\psi, \phi). \tag{2.1}$$

Next, we define the partial order on $\hat{E}_0 = E_0 \times E_0$ as follows:

For any $\xi_1, \xi_2 \in \hat{E}_0$, $\xi_1 = \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}$, $\xi_1 \le \xi_2$ iff $\phi_1 \le \phi_2$ and $\psi_1 \ge \psi_2$, $\phi_1, \phi_2, \psi_1, \psi_2 \in E_0$. Also, the metric on $\hat{E}_0$ is

$$\hat{d}_0(\xi_1, \xi_2) = \begin{bmatrix} d_0(\phi_1, \phi_2) \\ d_0(\psi_1, \psi_2) \end{bmatrix}.$$

We note that the same partial order holds for $\hat{E} = E \times E$ and the corresponding metric will be $\hat{d}_0$ replaced by $\hat{d}$ suitably.

Set $\xi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ and $\eta = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$. We have $P\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$, where $\phi, \psi \in E_0$, and we define $S : \hat{E}_0 \to \hat{E}$ as

$$S\xi = S\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} F(\phi, \psi) \\ F(\psi, \phi) \end{pmatrix} \tag{2.2}$$

where $\hat{E}_0 = E_0 \times E_0$ and $\hat{E} = E \times E$.

Since $F$ is continuous, $S$ is continuous.

To show that $S$ is monotone nondecreasing, consider $\xi_1, \xi_2 \in \hat{E}_0$, that is, $\xi_1 = \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}$, then $\xi_1 \le \xi_2$ implies $\phi_1 \le \phi_2$ and $\psi_1 \ge \psi_2$.

Consider

$$S\xi_1 = \begin{bmatrix} F(\phi_1, \psi_1) \\ F(\psi_1, \phi_1) \end{bmatrix}, \qquad S\xi_2 = \begin{bmatrix} F(\phi_2, \psi_2) \\ F(\psi_2, \phi_2) \end{bmatrix}.$$