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Classification of Hamiltonian-stationary Lagrangian submanifolds of constant curvature in CP^3 with positive relative nullity[☆]

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Abstract

A Lagrangian submanifold in a Kaehler manifold is said to be Hamiltonian-stationary if it is a critical point of the area functional restricted to (compactly supported) Hamiltonian variations. In this paper we classify Hamiltonian-stationary Lagrangian submanifolds of constant curvature in CP^3 with positive relative nullity. As an immediate by-product, several explicit new families of Hamiltonian-stationary Lagrangian submanifolds in CP^3 are obtained.

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1. Introduction

Let $\tilde{M}^n(4\tilde{c})$ denote a complex n-dimensional complex space of constant holomorphic sectional curvature $4\tilde{c}$. Let J be the complex structure and the Kaehler metric $\langle\ ,\ \rangle$ of $\tilde{M}^n(4\tilde{c})$. The Kaehler 2-form ω is defined by $\omega(\cdot,\cdot) = \langle J\cdot,\cdot\rangle$. An immersion $\psi : M \to \tilde{M}^n(4\tilde{c})$ of an n-manifold M into $\tilde{M}^n(4\tilde{c})$ is called *Lagrangian* if $\psi^*\omega = 0$ on M. A vector field X on $\tilde{M}^n(4\tilde{c})$ is called Hamiltonian if $\mathcal{L}_X\omega = f\omega$ for some function $f \in C^\infty(\tilde{M}^n(4\tilde{c}))$, where $\mathcal{L}$ is the Lie derivative. Thus, there exists a smooth real-valued function φ on $\tilde{M}^n(4\tilde{c})$ such that $X = J\tilde{\nabla}\varphi$, where $\tilde{\nabla}$ is the gradient in $\tilde{M}^n(4\tilde{c})$. The diffeomorphisms of the flux ψ_t of X satisfy $\psi_t\omega = \mathrm{e}^{h_t}\omega$. Thus they transform Lagrangian submanifolds into Lagrangian submanifolds.

Oh [16] studied the following variational problem: a normal vector field ξ to a Lagrangian immersion $\psi : M^n \to \tilde{M}^n(4\tilde{c})$ is called Hamiltonian if $\xi = J\nabla f$, where f is a smooth function on M^n and ∇f is the gradient of f with respect to the induced metric.

If $f \in C_0^\infty(M)$ and $\psi_t : M \to \tilde{M}^n(4\tilde{c})$ is a variation of ψ with $\psi_0 = \psi$ and variational vector field ξ, then the first variation of the volume functional is

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathrm{vol}(M, \psi_t^* g) = -\int_M (f\,\mathrm{div} JH)\mathrm{d}v_M,$$

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where H is the mean curvature vector of the immersion ψ and div is the divergence operator on M. Critical points of this variational functional are called *Hamiltonian-stationary* (or simply *H-stationary*). Lagrangian submanifolds with parallel mean curvature vector are always H-stationary. Among others, H-stationary Lagrangian submanifolds in complex space forms have been studied in [1–5,7,8,13–16].

The motivations for studying H-stationary Lagrangian submanifolds were due to their interesting geometric properties and the similarities to some models in incompressible elasticity.

Let h denote the second fundamental form of Lagrangian submanifold M of $\tilde{M}^n(4c)$. At a given point $p \in M$, the *relative null space* $\mathcal{N}_p$ at p is the subspace of the tangent space T_pM defined by

$$\mathcal{N}_p = \{X \in T_pM : h(X, Y) = 0\,\forall Y \in T_pM\}.$$

The dimension of $\mathcal{N}_p$ is called the *relative nullity* at p.

In this article, we completely classify the family of H-stationary Lagrangian submanifolds of constant sectional curvature in CP^3 with positive relative nullity. More precisely, we prove that there exist five families of H-stationary Lagrangian submanifolds of constant curvature in CP^3 with positive relative nullity. Conversely, every H-stationary Lagrangian submanifold of constant curvature in CP^3 with positive relative nullity is locally congruent to an open portion of a Lagrangian submanifold given by the five families.

2. Preliminaries

2.1. Basic notation and formulas

Let $f : M \to \tilde{M}^n(4\tilde{c})$ be a Lagrangian immersion of a Riemannian n-manifold M into $M^n(4\tilde{c})$. Denote the Riemannian connections of M and $M^n(4\tilde{c})$ by ∇ and $\tilde{\nabla}$, respectively; and by D the connection on the normal bundle of the submanifold.

The formulas of Gauss and Weingarten are (cf. [6])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for tangent vector fields X, Y and normal vector field ξ. If we denote the Riemann curvature tensor of ∇ by R, then the equations of Gauss and Codazzi are given respectively by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(X, Z), h(Y, W)\rangle + \tilde{c}\{\langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, W\rangle\}, \tag{2.3}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{2.4}$$

where $(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

For a Lagrangian submanifold M of $\mathbf{C}^n$, we have (cf. [11])

$$D_X JY = J\nabla_X Y, \tag{2.5}$$

$$\langle h(X, Y), JZ\rangle = \langle h(Y, Z), JX\rangle = \langle h(Z, X), JY\rangle\,. \tag{2.6}$$

2.2. Lagrangian and Legendrian submanifolds

We recall a basic relationship between Legendrian submanifolds of $S^{2n+1}(1)$ and Lagrangian submanifolds of the complex projective n-space CP^n with constant holomorphic curvature 4 (cf. [17]).

Let

$$S^{2n+1}(1) = \{(z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1} : \langle z, z\rangle = 1\}$$

be the unit hypersphere in $\mathbf{C}^{n+1}$ centered at the origin. On $\mathbf{C}^{n+1}$ we consider the complex structure J induced by $\mathrm{i} = \sqrt{-1}$. On $S^{2n+1}(1)$ we consider the canonical Sasakian structure consisting of ϕ given by the projection of the complex structure J of $\mathbf{C}^{n+1}$ on the tangent bundle of $S^{2n+1}(1)$ and the structure vector field $\xi = Jx$ with x being the position vector.

An isometric immersion $f : M \to S^{2n+1}(1)$ is called *Legendrian* if ξ is normal to $f_*(TM)$ and $\langle \phi(f_*(TM)), f_*(TM)\rangle = 0$, where $\langle\, ,\, \rangle$ denotes the inner product on $\mathbf{C}^{n+1}$. The vectors of $S^{2n+1}(1)$ normal to ξ at a point z