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Local well-posedness and quantitative ill-posedness for the Ostrovsky equation

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ABSTRACT

In this article we consider the initial value problem (IVP) for the Ostrovsky equation:

$$\begin{aligned}&\partial_t u - \partial_x^3 u \mp \partial_x^{-1} u + u\partial_x u = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R},\\ &u(x,0) = u_0(x),\end{aligned}$$

with initial data in Sobolev spaces $H^s(\mathbb{R})$. We prove that if $s > -\frac{3}{4}$ this IVP is locally well-posed in $H^s(\mathbb{R})$ and if $s < -\frac{3}{4}$ the IVP is not quantitatively well-posed in $H^s(\mathbb{R})$.

1. Introduction

In this article we consider the initial value problem (IVP) for the Ostrovsky equation:

$$\begin{aligned}&\partial_t u - \partial_x^3 u \mp \partial_x^{-1} u + u\partial_x u = 0, \quad x \in \mathbb{R}, t \in \mathbb{R},\\ &u(x,0) = u_0(x),\end{aligned} \tag{1.1}$$

with initial data u_0 in Sobolev spaces with negative indices. The operator ∂_x^{-1} in the equation denotes a certain antiderivative with respect to the variable x defined through the Fourier transform by $(\partial_x^{-1} f)^\frown(\xi) = \frac{\widehat{f}(\xi)}{i\xi}$. For the equation with the "−" sign in the third term we will refer to $(O1)$, and for the equation with the "+" sign we will refer to $(O2)$. This equation is a perturbation of the well-known Korteweg de Vries (KdV) equation with a nonlocal term and was deducted in [12] as a model for weakly nonlinear long waves, in a rotating frame of reference, to describe the propagation of surface waves in the ocean. The sign in the third term of the equation is related to the type of dispersion.

Linares and Milanés in [11] proved that the IVP (1.1) for both equations is locally well-posed for initial data u_0 in Sobolev spaces $H^s \equiv H^s(\mathbb{R})$, with $s > \frac{3}{4}$, and such that $\partial_x^{-1} u_0 \in L^2(\mathbb{R})$. This result was obtained by the use of certain regularizing effects of the linear part of the equation. In [8] we proved local well-posedness in Sobolev spaces H^s, with $s > -\frac{1}{2}$ for $(O1)$, and with $s > -\frac{3}{4}$ for $(O2)$. For that we used Bourgain spaces and the technique of elementary calculus inequalities, introduced by Kenig, Ponce, and Vega in [10]. Later, in [9], using the "I method" of [5], we proved the global well-posedness of the IVP (1.1) for both equations in H^s with $s > -\frac{3}{10}$. Besides, for $(O1)$ we improved to $s \geq -\frac{1}{2}$ the local result. Recently, Gui and Liu in [6], established for $(O1)$ the local well-posedness of the IVP (1.1) in H^s with $s \geq -\frac{7}{12}$.

In this paper we prove that the IVP (1.1) for $(O1)$ is locally well-posed in $(H^s, \|\cdot\|_s)$ with $s > -\frac{3}{4}$, obtaining the same result we already had for $(O2)$, and getting closer to the optimal result ($s \geq -\frac{3}{4}$) of local well-posedness for the KdV equation [10,3].

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To prove our result we combine elementary calculus inequalities with a local smoothing effect of Kato type and a maximal-type estimate for certain multiplier operator, as it was done in [4].

On the other hand, using the abstract framework described by Bejenaru and Tao in [2], we show that the IVP (1.1) is not "quantitatively" well-posed in H^s with $s < -\frac{3}{4}$. This is achieved as in [1], by studying the third Picard iteration of the integral problem associated to the IVP (1.1). We will search for the solutions of problem (1.1) in Bourgain spaces $X_{s,b}$ defined by

$$X_{s,b} := \left\{ u \in \mathcal{S}'(\mathbb{R}^2) \mid \|u\|_{s,b}^2 := \int_{\mathbb{R}^2} \langle\xi\rangle^{2s}\langle\sigma\rangle^{2b}|\widehat{u}(\lambda)|^2 \mathrm{d}\lambda < \infty \right\},$$

where $s, b \in \mathbb{R}$, $\mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions in $\mathbb{R}^2$, $\widehat{u}$ is the Fourier transform of u with respect to the space and time variables, $\lambda := (\xi, \tau)$ is the variable in the frequency space with ξ corresponding to the space variable x, τ corresponding to the time variable t, and $\sigma := \tau - m(\xi) \equiv \tau + (\xi^3 \pm \frac{1}{\xi})$ is the symbol associated to the linear part of the Ostrovsky equation, with "+" for (O1) and "−" for (O2).

We also will consider, for $s \in \mathbb{R}$, the space

$$X_s := \{u \in \mathcal{S}'(\mathbb{R}^2) \mid \|u\|_{X_s} := \|\langle\xi\rangle^s \widehat{u}(\lambda)\|_{L^2_\xi L^1_\tau} < \infty\}.$$

For $b > \frac{1}{2}$ the space $X_{s,b}$ is continuously embedded in X_s, ($X_{s,b} \hookrightarrow X_s$), and the space X_s is continuously embedded in $C_{bd}(\mathbb{R}_t; H^s)$, the space of continuous and bounded functions of the time variable t with values in H^s. In this manner, for $b > \frac{1}{2}$ and $T > 0$ we can consider the space $X_{s,b}[-T, T]$ of the restrictions to $[-T, T]$ of the elements in $X_{s,b}$.

Our concept of solution for the IVP (1.1) comes from Duhamel's formula: formally, $u \in X_{s,b}[-T, T]$ is a solution of the IVP (1.1) in $[-T, T]$ if and only if for all $t \in [-T, T]$

$$u(t) = W(t)u_0 - \frac{1}{2}\int_0^t W(t-t')\partial_x(u(t'))^2 \mathrm{d}t', \tag{1.2}$$

where $W(t)u_0$ is the solution of the linear problem associated to (1.1), that is

$$[W(t)u_0]^\frown(\xi) = \mathrm{e}^{\mathrm{i}tm(\xi)}\widehat{u_0}(\xi).$$

To obtain local solutions of the IVP (1.1) we consider three operators L, G_T and B associated to the right side of (1.2). The construction of these operators is as follows:

For $u_0 \in H^s$ let

$$L(u_0) := \psi(\cdot_t)W(\cdot_t)u_0, \tag{1.3}$$

where $\psi \in C_0^\infty(\mathbb{R}_t)$, $0 \le \psi \le 1$, $\psi \equiv 1$ in $[-1, 1]$ and supp $\psi \subset (-2, 2)$.

From a direct calculation it may be proved that

$$\|L(u_0)\|_{s,b} \le C\|u_0\|_s, \tag{1.4}$$

and thus $L : H^s \longrightarrow X_{s,b}$ is a continuous linear operator.

We define G_T as a modification of the integral operator in Duhamel's formula (for a detailed definition of G_T see [7]). This modification is defined in such a way that for $f \in \mathcal{S}(\mathbb{R}^2)$, $0 < T \le 1$ and $t \in [0, T]$

$$G_T(f)(t) = -\frac{1}{2}\int_0^t W(t-t')f(t')\mathrm{d}t'. \tag{1.5}$$

Proceeding as it was done in [7], Lemma 1, it may be proved that for $s \in \mathbb{R}$, $b \in (\frac{1}{2}, 1)$, and $\beta \in (0, 1-b)$, there is $\delta > 0$, such that for $f \in \mathcal{S}(\mathbb{R}^2)$

$$\|G_T(f)\|_{s,b} \le CT^\delta\|f\|_{s,-\beta} \quad \forall T \in (0, 1). \tag{1.6}$$

In a similar manner it may be seen that for $f \in \mathcal{S}(\mathbb{R}^2)$

$$\|G_T(f)\|_{X_s} \le C\|f\|_{X_s} \quad \forall s \in \mathbb{R},$$

allowing us to extend G_T to a continuous operator from X_s to X_s.

If we consider the space $X := \cap_{s\in\mathbb{R}} X_s$, then G_T sends X in X. Besides, since $\mathcal{S}(\mathbb{R}^2)$ is dense in X_s for all $s \in \mathbb{R}$ and $X_s \hookrightarrow C_{bd}(\mathbb{R}_t; H^s)$, it follows that equality (1.5) is still valid for $f \in X$.

Using the fact that X is an algebra, closed under the differentiation ∂_x, we define $B : X \times X \longrightarrow X$ by

$$B(u, v) = G_T(\partial_x(uv)). \tag{1.7}$$

For the bilinear form $\partial_x(uv)$, associated to the nonlinear term of the equation, in Section 2 we will prove the following lemma.

Lemma 1. *In the case of (O1), for $s > -\frac{3}{4}$, if $r := \max\{-s, 0\}$, $\beta \in (\frac{5}{12}, \frac{1}{2})$ is chosen in such a way that $(\frac{1}{2} - \beta) \le \frac{1}{3}(\frac{3}{4} - r)$, and $b > \frac{1}{2}$ is chosen to satisfy $\frac{1}{2}(\frac{1}{2} - \beta) \le b - \frac{1}{2}$ and $b + \beta < 1$, then*

$$\|\partial_x(uv)\|_{s,-\beta} \le C\|u\|_{s,b}\|v\|_{s,b} \quad \forall u, \ v \in X \cap X_{s,b}. \tag{1.8}$$