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A class of semilinear parabolic equations with singular initial data

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ABSTRACT

We consider the initial-boundary value problem for the semilinear parabolic equation on a smooth domain $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} u_t = \Delta u + |\nabla u|^p |u|^{q-1} u & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $1 \le p \le 2$ and $q \ge 1$. In this paper, we are concerned with the existence of solutions with singular initial data $u_0 \notin L^\infty$. We study the problem (1.1) on several singular spaces of initial data. More precisely, we investigate the subquadratic case $p < 2$ in the Lebesgue class $\{L^r\}_{1\le r<\infty}$ and in the singular Sobolev class $\{W_0^{1,r}\}_{1\le r<N}$ for $N \ge 2$. Moreover, in the quadratic case $p = 2$, we present some evidence that local existence may fail in the case of the critical Sobolev space $W_0^{1,N}$, when $N \ge 2$.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ or the whole space $\mathbb{R}^N$. We consider the initial-boundary value problem for the semilinear parabolic equation of the following form:

$$\begin{cases} u_t = \Delta u + |\nabla u|^p |u|^{q-1} u & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $1 \le p \le 2$ and $q \ge 1$.

It is well known that the semilinear parabolic equation $u_t = \Delta u + F(u, \nabla u)$ with a locally Lipschitz function $F : \mathbb{R}^{N+1} \to \mathbb{R}$ has a local-in-time unique solvability in a sufficiently regular class. Indeed, it can be shown that if $u_0 \in W_0^{1,\infty}(\Omega)$, then the corresponding problem has a unique local solution $u \in L^\infty([0,T), W_0^{1,\infty}(\Omega)) \cap C^{1,2}((0,T)\times\overline{\Omega})$ for some $T > 0$, which is a classical solution on $(0,T)\times\overline{\Omega}$ with $\|u(t) - S(t)u_0\|_{W^{1,\infty}} \to 0$ as $t \downarrow 0$. Here $(S(t))_{t\ge0}$ is the heat semigroup with the homogeneous Dirichlet boundary condition (see Eq. (2.2)).

In this paper, we are concerned with the existence of solutions of (1.1) with singular initial data $u_0 \notin L^\infty$. Now let us review some known results related to our investigation.

The landmark work in this direction has been done by Weissler [10,11], who establish the local L^r theory of the semilinear parabolic equation with the power nonlinearity $|u|^{q-1}u$,

$$\begin{cases} u_t = \Delta u + |u|^{q-1} u & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

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for a given $u_0 \in L^r(\Omega)$, $1 \le r < \infty$. The following existence and uniqueness result is established, where the uniqueness part is due to Brezis and Cazenave [5].

Theorem 1.1 *([10,11,5]). Put $r_0 = \frac{N(q-1)}{2}$. Assume $r > r_0$ (resp. $r = r_0$) and $r \ge 1$ (resp. $r > 1$). Given $u_0 \in L^r(\Omega)$, there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], L^r(\Omega)) \cap C^{1,2}((0,T) \times \overline{\Omega})$ with $u(0) = u_0$, which is a classical solution of (1.2) on $(0,T) \times \overline{\Omega}$.*

On the other hand, Ben-Artzi, Souplet and Weissler [2] recently obtain very elaborate results for the semilinear parabolic equation with the gradient nonlinearity $|\nabla u|^p$ (which is referred as the viscous Hamilton–Jacobi equation),

$$\begin{cases} u_t = \Delta u + |\nabla u|^p & \text{in } (0,\infty) \times \Omega, \\ u(t,x) = 0 & \text{in } (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.3}$$

They provide the local theories for the problem (1.3) on L^r for $1 \le r < \infty$ in the subquadratic case $1 \le p < 2$ and also on the Sobolev space $W_0^{1,r}$ for $1 \le r < \infty$ in the general case $p \ge 1$ as follows.

Theorem 1.2 *([2]). Let $1 \le p < 2$ and let $r_1 = \frac{N(p-1)}{2-p}$. Assume $r > r_1$ (resp. $r = r_1$) and $r \ge 1$ (resp. $r > 1$). Given $u_0 \in L^r(\Omega)$, there exists a unique function $u \in C([0,\infty), L^r(\Omega)) \cap C^{1,2}((0,\infty) \times \overline{\Omega})$ with $u(0) = u_0$, which is a classical solution of (1.3) on $(0,\infty) \times \overline{\Omega}$.*

Theorem 1.3 *([2]). Let $r_2 = N(p-1)$. Assume $r > r_2$ (resp. $r = r_2$) and $r \ge 1$ (resp. $r > 1$). Given $u_0 \in W_0^{1,r}(\Omega)$, there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], W_0^{1,r}) \cap C^{1,2}((0,T) \times \overline{\Omega})$ with $u(0) = u_0$, which is a classical solution of (1.3) on $(0,T) \times \overline{\Omega}$.*

Remark 1.4. When $p \le 2$, all solutions of (1.3) exist globally. This follows from the facts that the maximum principle gives a uniform bound in u and the nonlinearity satisfies Bernstein's quadratic condition (cf. [3,7]). On the contrary, if $p > 2$, then the gradient blowup phenomenon may occur in finite time $T > 0$ (cf. [9]). In addition, if a domain Ω is replaced by $\mathbb{R}^N$ or a compact Riemannian manifold M without boundary, the gradient blowup does not occur even for every $p > 2$, that is, (1.3) always admits global smooth solutions (cf. [1]).

Remark 1.5. In the framework of the space of continuous functions, the unique solvability for (1.3) is obtained for general $p > 0$. Indeed, it is shown in [6] when $\Omega = \mathbb{R}^N$ that for every initial data $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the Cauchy problem (1.3) admits a unique classical solution

$$u \in C([0,\infty) \times \mathbb{R}^N) \cap L^\infty([0,\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,\infty) \times \mathbb{R}^N).$$

In Theorem 1.3, we note that when $p > 2$, the space of initial data $W_0^{1,r}(\Omega)$ is contained in $L^\infty(\Omega)$, since $r \ge r_2 > N$. Therefore, one can show similarly that the problem (1.1) has a local solution $u \in C([0,T], W_0^{1,r}(\Omega))$ for every $u_0 \in W_0^{1,r}(\Omega)$, $r \ge N(p-1)$, provided that $p > 2$. This is the reason why we have restricted our attention to the case when $p \le 2$ in the problem (1.1).

In this paper, we establish the existence of solutions in the class $C([0,T], E)$ for the initial-boundary problem (1.1), where we take a Banach space E in the Lebesgue class $\{L^r\}_{1 \le r < \infty}$ and in the singular Sobolev class $\{W_0^{1,r}\}_{1 \le r < N}$. Moreover, in the quadratic case of $p = 2$, we present some evidence that local existence of (1.1) does not hold on $W_0^{1,N}$ in a reasonable class of solutions.

Roughly speaking, our main results are summarized as follows.

1. When $p < 2$, existence holds on L^r for $r \ge \frac{N(p+q-1)}{2-p}$. (cf. Theorem 2.1)
2. When $N \ge 2$ and $p < 2$, existence holds on $W_0^{1,r}$ for $r \in \left[\frac{N(p+q-1)}{q+1}, N\right)$. (cf. Theorem 3.1)
3. When $N \ge 2$ and $p = 2$, existence fails on $W_0^{1,N}$. (cf. Theorem 4.1)

The outline of this paper is as follows. In Section 2, we establish existence and uniqueness in the Lebesgue class for $p < 2$. In Section 3 we focus on (1.1) in the singular Sobolev class for $p < 2$. Finally, in Section 4 we provide some nonexistence result on $W_0^{1,N}$ in the quadratic case $p = 2$.

2. In the Lebesgue class

In this section, we consider the problem (1.1) in the Lebesgue class $\{L^r\}_{1 \le r < \infty}$ when $1 \le p < 2$.

Theorem 2.1. *Let $1 \le p < 2$ and let $\rho_0 = \frac{N(p+q-1)}{2-p}$. Assume $r = \rho_0 > 1$ (resp. $r > \rho_0$ with $r \ge 1$). Given $u_0 \in L^r(\Omega)$, there exist a unique function $u \in C([0,\infty), L^r(\Omega)) \cap C^{1,2}((0,\infty) \times \overline{\Omega})$ with $u(0) = u_0$, which is a classical solution of (1.1) on $(0,\infty) \times \overline{\Omega}$.*