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Best approximation theorems for nonexpansive and condensing mappings in hyperconvex spaces

Jack Markin [a,*], Naseer Shahzad [b]

[a] 528 Rover Blvd., Los Alamos, NM 87544, United States
[b] Department of Mathematics, King Abdul Aziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

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ABSTRACT

In hyperconvex metric spaces we consider best approximation, invariant approximation and best proximity pair problems for multivalued mappings that are condensing or nonexpansive.

1. Introduction

Originated by Ky Fan in [1] for normed linear spaces, the best approximation problem has been extensively studied in linear spaces and more recently in hyperconvex metric spaces. For our purpose best approximation problems in a hyperconvex space consist of finding conditions for a given multivalued mapping F and set A such that there is a point $x \in A$ satisfying $d(x, F(x)) \leqslant d(y, F(x))$ for $y \in A$. Best approximation theorems for mappings in hyperconvex metric spaces are given for the point-valued case in [2,3] and for the multivalued case in [4–6], where in both cases the mappings are assumed to be either continuous or nonexpansive. A best approximation theorem in a complete R-tree appeared in [7] for continuous point-valued mappings. Here we extend the previous multivalued results in hyperconvex spaces to include condensing mappings.

The invariant approximation problem consists of finding conditions for a multivalued mapping F with invariant set A and nonempty fixed point set $Fix(F)$ (with $p \in Fix(F)$), implying that $Fix(F) \cap P_A(p) \neq \emptyset$, where P_A is the metric projection onto A. Invariant approximation problems have been studied in normed linear spaces for point-valued mappings [8–10] and for multivalued mappings in [11,12]. Considering multivalued nonexpansive and condensing mappings in hyperconvex spaces, we obtain analogues of the linear space results.

Given subsets A, B and a mapping $F : A \to 2^B$ the best proximity pair problem consists of finding conditions on F, A and B implying that there is a point $x \in A$ such that $d(x, F(x)) = inf\{d(a, b) : a, b \in A, B\}$. Best proximity pair results for normed

* Corresponding author. Tel.: +1 505 672 9102.
E-mail addresses: jmarkin@newmexico.com (J. Markin), nshahzad@kau.edu.sa (N. Shahzad).

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linear spaces appeared in [13,14], for hyperconvex spaces in [15,16] and for CAT(0) spaces in [17]. We extend some results for nonexpansive and continuous multivalued mappings obtained in [16] to the case of condensing multivalued mappings.

2. Definitions

A metric space (M,d) is said to be *hyperconvex* if and only if for any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta$, it is the case that

$$\cap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset,$$

where $B(x, r)$ denotes the closed ball with center $x \in M$ and radius r.

A hyperconvex space is a nonexpansive retract of any metric space in which it is isometrically imbedded. Hyperconvex metric spaces were introduced and their basic properties elaborated in [18]. We denote the family of nonempty subsets of M by 2^M and the nonempty bounded subsets of M by $B(M)$.

The *admissible* subsets of a hyperconvex space M are sets of the form $\cap_\alpha B(x_\alpha, r_\alpha)$, that is, the family of ball intersections in M.

For a subset S of M, $N_\varepsilon(S)$ denotes the closed ε-neighborhood of S, that is, $N_\varepsilon(S) = \{x \in M : d(x, S) \leqslant \varepsilon\}$, where $d(x, S) = inf_{y\varepsilon S} d(x, y)$. If S is admissible, then $N_\varepsilon(S)$ is admissible [19].

A subset S of a metric space M is said to be *externally hyperconvex* if given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leqslant r_\alpha + r_\beta \quad \text{and} \quad d(x_\alpha, S) \leqslant r_\alpha$$

it follows that $\cap_\alpha B(x_\alpha, r_\alpha) \cap S \neq \emptyset$.

If the set S is externally hyperconvex then $N_\varepsilon(S)$ is externally hyperconvex [20].

Every admissible set is externally hyperconvex, and the externally hyperconvex sets are proximinal in M, that is, if $x \in M$ and E is externally hyperconvex then there is an $e \in E$ such that $d(x, e) = d(x, E)$ (see for example [20]). The externally hyperconvex sets are proximinal nonexpansive retracts of M [21].

For any set X in M the *cover* of the set is defined by

$$cov(X) = \cap_\alpha \{B_\alpha \subset M : B_\alpha \text{ is a closed ball and } X \subset B_\alpha\}.$$

A set X is called *sub-admissible* if for each finite subset Y of X, $cov(Y) \subset X$.

For any pair of points $x, y \in M$ a *geodesic path* joining these points is a map c from a closed interval $[0, r]$ to M such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. The mapping c is an isometry and $d(x, y) = r$. The image of c is called a *geodesic segment* joining x and y which when unique is denoted by $[x, y]$. A *geodesic ray* in M is a subset of M isometric to the half-line $[0, \infty)$.

An *R-tree* is a metric space M such that:

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in M$;
(ii) if $x, y, z \in M$ then $[x, y] \cap [x, z] = [x, w]$ for some $w \in M$;
(iii) if $x, y, z \in M$ and $[x, y] \cap [y, z] = \{y\}$ then $[x, y] \cup [y, z] = [x, z]$.

A metric space is a complete R-tree if and only if it is hyperconvex and has unique geodesic segments [22]. A subset X of M is said to be *convex* if X includes every geodesic segment joining any two of its points. Any closed convex subset of an R-tree is a proximinal nonexpansive retract of the space.

If U, V are bounded subsets of M, let D be the *Hausdorff metric* defined as usual by $D(U, V) = inf\{\varepsilon > 0 : U \subset N_\varepsilon(V) \text{ and } V \subset N_\varepsilon(U)\}$. In a metric space M, a mapping $F : M \to 2^M$ with nonempty bounded values is *nonexpansive* provided $D(F(x), F(y)) \leqslant d(x, y)$. For a set X in a metric space M and any point $x \in M$ the set of best approximations to x in the set X is denoted by

$$P_X(x) = \{y \in X : d(x, y) = d(x, X)\}.$$

The *Kuratowski measure of noncompactness* $\phi : B(M) \to [0, \propto)$ is defined by

$$\phi(X) = inf\{\varepsilon > 0 : X \subset \cup_{k=1}^n X_k \text{ with } X_k \in B(M), diam(X_k) \leqslant \varepsilon\}.$$

A mapping $F : M \to B(M)$ is said to be *condensing* provided $\phi(F(X)) < \phi(X)$, for any $X \in B(M)$ with $\phi(X) > 0$, where $F(X) = \cup_{x \in X} F(x)$.

For A, B nonempty subsets of a metric space M, we define the sets

$$A_0 = \{x \in A : d(x, y) = dist(A, B) \text{ for some } y \in B\}$$
$$B_0 = \{x \in B : d(x, y) = dist(A, B) \text{ for some } y \in A\},$$

where $dist(A, B) = inf\{d(x, y) : x \in A \text{ and } y \in B\}$.

Given two subsets A, B of a metric space and a mapping $F : A \to 2^B$, if there is a point $x \in A$ such that $d(x, F(x)) = dist(A, B)$ then $(x, F(x))$ is called a *best proximity pair*.