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Existence of solutions for some nonlinear problems with p-Laplacian like operators

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ABSTRACT

The purpose of this paper is to obtain some existence results of solutions for the nonlinear boundary value problems with p-Laplacian like operators.

1. Introduction

Among the studies on equations, the nonlinear boundary value problems play a central role [1–11]. Recently, problems of periodic solutions for the p-Laplacian have became popular. Some works can be found in [12,13,15,16] and the references therein. In [12,15], the Manásevich–Mawhin continuation theorem had been stated and some existence results for the various boundary value problems were proved, which contain Dirichlet, periodic, Neuman problems and so on.

In this paper we study the existence of solutions for the nonlinear boundary value problems of the form

$$(\phi_p(u'))' = A(t)u + f(t, u, u'), \qquad au(0) - bv(0) = 0 = cu(T) + dv(T), \tag{1.1}$$

where $p > 1$, $\phi_p : \mathbb{R}^1 \to \mathbb{R}^1$ is an increasing homeomorphism which includes p-Laplacian like operator $\phi_p(s) = |s|^{p-2}s$, $\phi_p(0) = 0, f : [0, T] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1, A(t) \gneqq 0$ are continuous functions, $v = \phi_p(u')$, $a, b, c, d \geq 0$, and $a + b > 0$, $c + d > 0$.

The paper is organized as follows. In Section 2, we will introduce a valuable lemma. Section 3 is devoted to our main result on a continuation theorem. In Section 4 we give upper and lower solution results for (4.12), there we use the continuation theorem stated in Section 3.

2. Preliminaries

We first introduce some notations. Let

$$X = \left\{ \begin{array}{l} (u, v) \in C([0, T], \mathbb{R}^1) \times C([0, T], \mathbb{R}^1) : \\ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \begin{pmatrix} u(T) \\ v(T) \end{pmatrix} = 0 \end{array} \right\}$$

denote the Banach space, equipped with the norm $\|(u, v)\|_X = \max_{[0,T]} |u(s)| + \max_{[0,T]} |v(s)|$.

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We denote by ψ_p the inverse ϕ_p^{-1} such that $\phi_p \circ \psi_p = id$. It is clear that ψ_p is also an increasing homeomorphism. Thus problem (1.1) can be written in the equivalent form

$$u' = \psi_p(v),$$
$$v' = A(t)u + f(t, u, \psi_p(v))$$

with the corresponding boundary conditions.

Lemma 2.1. *For each* $\lambda \in [0, 1]$, *the boundary value problem*

$$\begin{aligned} &u' = \lambda\psi_p(v), \qquad v' = A(t)u, \\ &au(0) - bv(0) = 0 = cu(T) + dv(T) \end{aligned} \tag{2.2}$$

has a unique solution $(u, v) = (0, 0)$.

Proof. A. The case $\lambda = 0$. The problem (2.2) is equivalent to

$$\begin{aligned} &u' = 0, \qquad v' = A(t)u, \\ &au(0) - bv(0) = 0 = cu(T) + dv(T). \end{aligned} \tag{2.3}$$

Suppose that (2.3) has a nonzero solution (u, v). Then

$$u(t) \equiv \lambda_0, \qquad v(t) = v(0) + \lambda_0 \int_0^t A(s)\mathrm{d}s,$$

and

$$a\lambda_0 - bv(0) = 0,$$
$$c\lambda_0 + dv(T) = c\lambda_0 + dv(0) + d\lambda_0 \int_0^T A(s)\mathrm{d}s = 0.$$

Taking now $\lambda_0 > 0$ we deduce that $bv(0) \geq 0$.

If $a \neq 0$ then $bv(0) > 0$. This implies that $v(0) > 0$, so that $d\lambda_0 \int_0^T A(s)\mathrm{d}s = 0 = -c\lambda_0 - dv(0) < 0$. It is clear that $c + d \neq 0$ and $\int_0^T A(s)\mathrm{d}s > 0$, a contradiction. If $a = 0$ then $bv(0) = 0$. Using $v(0) = 0$, we deduce that $d\lambda_0 \int_0^T A(s)\mathrm{d}s = 0 = c\lambda_0$. From $c + d \neq 0$ and $\int_0^T A(s)\mathrm{d}s > 0$, we can obtain a contradiction.

B. The case $\lambda > 0$. Suppose that there exists $\lambda_1 \in (0, 1]$, such that the boundary value problem (2.2) has a nonzero solution (u, v). Then there will be the following four cases:

(a) $u(t)$ takes the positive maximum in $(0, T)$.

Let $t_0 \in (0, T)$ be such that $u(t_0) = \max_{[0,T]} u(t)$. Then $u'(t_0) = 0$. On the other hand, because $\lambda_1 > 0$, we see that $v(t_0) = 0$. Let $E = \{t \in [t_0, T] : u(s) > 0, s \in [t_0, t)\}$ and $\bar{t} = \sup E$. Using (2.2) it follows that

$$v(t) = v(t_0) + \int_{t_0}^t A(s)u(s)\mathrm{d}s = \int_{t_0}^t A(s)u(s)\mathrm{d}s \geq 0, \quad t \in [t_0, \bar{t}].$$

Then from (2.2) again, we obtain

$$u(t) = u(t_0) + \lambda_1 \int_{t_0}^t \psi_p(v(s))\mathrm{d}s \geq 0, \quad t \in [t_0, \bar{t}].$$

We now claim that $\bar{t} = T$. Suppose by contradiction that there exists $\delta > 0$ such that $u(t) > \frac{1}{2}u(t_0)$ with $t \in [\bar{t}, \bar{t} + \delta] \subset [\bar{t}, T]$, it follows that

$$v(t) = v(t_0) + \int_{t_0}^t A(s)u(s)\mathrm{d}s = \int_{t_0}^t A(s)u(s)\mathrm{d}s \geq 0, \quad t \in [t_0, \bar{t} + \delta],$$

and hence

$$u(t) = u(t_0) + \lambda_1 \int_{t_0}^t \psi_p(v(s))\mathrm{d}s \geq 0, \quad t \in [t_0, \bar{t} + \delta],$$

a contradiction. This implies that $u(T) = u(t_0)$, $v(T) \geq 0$ and $v(t) = u(T) \int_{t_0}^t A(s)\mathrm{d}s$ with $t \in [t_0, T]$.

Clearly, if $v(T) = 0$, then $A(t) \equiv 0$ with $t \in [t_0, T]$. Let $t_1 = \min\{t \in [0, t_0] : v(s) \equiv 0, u(s) \equiv u(t_0), s \in [t, T]\}$ such that

$$v(t) = \int_{t_0}^t A(s)u(s)\mathrm{d}s \leq 0, \quad t < t_0,\ u(s) \geq 0,\ s \in [t_1, t_0].$$

From $u' = \lambda_1\psi_p(v)$, we obtain

$$u(t) = u(t_0) + \lambda_1 \int_{t_0}^t \psi_p(v(s))\mathrm{d}s \geq x(t_0), \quad t < t_0,\ u(s) \geq 0,\ s \in [t_1, t_0].$$