Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 67 (2007) 103–125
Nonlinear Analysis
www.elsevier.com/locate/na

Weak solutions for nonlocal boundary value problems with low regularity data

Yuandi Wang*

Department of Mathematics, Shanghai University, Shanghai 200444, China

Received 7 April 2006; accepted 1 May 2006

Abstract

This paper concerns the existence and uniqueness of weak solutions for elliptic and parabolic equations under nonlocal boundary conditions, based on maximal regularity. It also gives the positivity of solutions which can be used in monotone iteration methods. As an application, the results are used to discuss some specific nonlocal problems.

MSC: 35J25; 35J65; 35K20; 35K55; 35K57

Keywords: Existence; Maximal regularity; Nonlocal; Boundary condition

1. Introduction

We consider the following quasilinear parabolic initial boundary value problem (IBVP for short):

$$\begin{cases} u_t + \mathcal{A}(t,x,u)u = f(t,x,u), & \text{in } Q_T, \\ \mathcal{B}(t,x,u)u = \kappa(t,x,u), & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), & \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) lying on one side of its boundary $\Gamma = \partial\Omega$, $\Gamma = \Gamma_0 \cup \Gamma_1$ is a C^2-submanifold with $\Gamma_0 \cap \Gamma_1 = \emptyset$. $Q_T := (0,T) \times \Omega$.

$$\mathcal{A}(t,x,u)v = -\nabla(\mathbf{a}(t,x,u)\nabla v + \vec{b}(t,x,u)v) + \vec{c}(t,x,u) \cdot \nabla v + a_0(t,x,u)v$$

* Tel.: +86 21 66133906; fax: +86 21 66133292.
E-mail address: yuandi_wang@yahoo.com.cn.

0362-546X/$ - see front matter
doi:10.1016/j.na.2006.05.002

and $\mathcal{A}(t, x, u)$ is strongly elliptic. The boundary operator is given by

$$\mathcal{B}(t, x, u)v := \delta\{\partial_{\nu_{\mathbf{a}}} v + (\gamma\vec{b}\cdot\vec{\nu} + d)\gamma v\} + (1-\delta)\gamma v.$$

The coefficients $\mathbf{a} = (a_{ij})_{n\times n}$, $\vec{b}$, $\vec{c}$, a_0 and d satisfy regularity conditions on $\bar{Q}_T \times \mathbb{R}$ or on $[0, T] \times \Gamma \times \mathbb{R}$, respectively specified below. $\partial_{\nu_{\mathbf{a}}} u := \gamma \mathbf{a}\nabla u \cdot \nu$, ν is the outer unit-normal vector on Γ, $\delta : \Gamma \to \{0, 1\}$ is defined by $\delta^{-1}(j) := \Gamma_j$ for $j = 0, 1$, γ denotes the trace operator.

We assume that

$$f : \bar{Q}_T \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \kappa : [0, T] \times \Gamma \times \mathbb{R} \to \mathbb{R}$$

($\mathbb{R}^+ := [0, +\infty)$) are Carathéodory functions; that is, f (respectively, κ) is measurable in $(t, x) \in \bar{Q}_T$ (respectively, in $(t, x) \in [0, T] \times \Gamma$) for each $u \in \mathbb{R}$ and continuous in u for a.e. $(t, x) \in \bar{Q}_T$ (respectively, $(t, x) \in [0, T] \times \Gamma$). We also consider the case where κ is a nonlocal function, e.g., $\kappa(t, x, u) = \int_\Omega k(t, x, y, u(t, y))\mathrm{d}y$.

The following special case of problem (1.1) has been treated in [13,19,20], for example, within the framework of classical solutions by means of the monotone iteration technique:

$$\begin{cases} u_t + \mathcal{A}_1 u = f(x, u), & \text{in } Q_T, \\ \dfrac{\partial u}{\partial \nu} + bu = \displaystyle\int_\Omega k_1(x, y)u(t, y)\mathrm{d}y, & \text{on } \partial\Omega, \\ u(x, 0) = u_0(x), & \text{on } \Omega \end{cases} \tag{1.2}$$

where b is a nonnegative constant, $k_1(x, y) \geq 0$ or $\int_\Omega |k_1(x, y)|\mathrm{d}y < 1$, $\mathcal{A}_1(x)$ is a linear strongly elliptic operator, and all data are assumed to be sufficiently smooth. The nonlocal IBVP (1.2) stands, e.g., for a model problem arising from quasistatic thermoelasticity. Results on linear problems can be found in [13,16,18], etc. As far as we know, this kind of boundary condition appeared first in 1952 in a paper [15] by W. Feller who discussed the existence of semi-groups. There are other problems leading to this boundary condition, e.g., control theory, where $\int_\Omega k_1(x, y)u(t, y)\mathrm{d}y = \sum_{j=1}^{J} w_j(x)\int_\Omega \tilde{k}_j(y)u(t, y)\mathrm{d}y$, see [17,1] etc. Some other fields such as environmental science [9] and chemical diffusion [21] also give rise to such kinds of problems. We do not give further comments here.

It is our purpose to study weak (or very weak) solutions of the corresponding stationary and evolution problem. We divide the paper into two sections. In the next section the elliptic problem is studied. It includes existence, and positivity of weak and very weak solutions. Semilinear parabolic problems are treated in the last section. Linear and quasilinear problems are investigated there by using maximal regularity. The results are then applied to a semilinear equation. In particular, we improve in this paper the known results on positivity (see Theorem 3.11).

2. Elliptic problem

Let $\Omega_1 := \Omega \cup \Gamma_1$, $\mathbb{R}^{n\times n}_{\text{post}}$ is the set of real symmetric and positive definite $(n\times n)$-matrices, and

$$\mathbb{E}(\Omega) := C^1(\bar{\Omega}, \mathbb{R}^{n\times n}_{\text{post}}) \times (C^1(\bar{\Omega}, \mathbb{R}^n))^2 \times C(\bar{\Omega}, \mathbb{R}) \times C(\partial\Omega, \mathbb{R}).$$

We set

$$\begin{cases} \mathcal{A}u := -\nabla(\mathbf{a}\nabla u + \vec{b}u) + \vec{c}\cdot\nabla u + a_0 u, & \text{in } \Omega, \\ \mathcal{B}u := \delta\{\partial_{\nu_{\mathbf{a}}} u + (\gamma\vec{b}\cdot\vec{\nu} + d)\gamma u\} + (1-\delta)\gamma u, & \text{on } \Gamma \end{cases} \tag{2.1}$$

where $\mathbf{a} := (a(x)_{ij})_{n\times n}$, and $(\mathbf{a}, \vec{b}, \vec{c}, a_0, d) \in \mathbb{E}(\Omega)$.