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Existence of exact penalty for constrained optimization problems in Hilbert spaces

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Abstract

In this paper we use the penalty approach in order to study two constrained minimization problems in Hilbert spaces. A penalty function is said to have the exact penalty property if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem. In this paper we establish a sufficient condition for the exact penalty property.

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Keywords: Critical point; Frechet differentiable function; Hilbert space; Minimization problem; Penalty function

1. Introduction and the main results

Penalty methods are an important and useful tool in constrained optimization. See, for example, [1,2,4–6,8–13,15–19] and the references mentioned there. In this paper we use the penalty approach in order to study two constrained nonconvex minimization problems with smooth cost functions. The first problem is an equality-constrained problem in a Hilbert space with a smooth constraint function and the second problem is an inequality-constrained problem in a Hilbert space with a smooth constraint function. A penalty function is said to have the exact penalty property[1,2,8,10] if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem. The notion of exact penalization was introduced by Eremin [9] and Zangwill [18] for use in the development of algorithms for nonlinear constrained optimization. For a detailed historical

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review of the literature on exact penalization, see Boukari and Fiacco [1], Burke [2] and Di Pillo and Grippo [8]. In this paper we study the existence of exact penalty. Note that the classes of minimization problems with a smooth objective function and a smooth constraint are considered in the book by Cesari on optimal control [3].

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|x\| = \langle x, x \rangle^{1/2}$, $x \in X$. Denote by $C^1(X; R^1)$ the set of all Frechet differentiable functions $f : X \to R^1$ such that the mapping $x \to f'(x)$, $x \in X$ is continuous. Here $f'(x) \in X$ is a Frechet derivative of f at $x \in X$.

Denote by $C^2(X; R^1)$ the set of all Frechet differentiable functions $f \in C^1(X; R^1)$ such that the mapping $x \to f'(x)$, $x \in X$ is also Frechet differentiable and that the mapping $x \to f''(x)$, $x \in X$ is continuous. Here $f''(x)$ is a Frechet second-order derivative of f at $x \in X$. It is a linear continuous self-mapping of X.

For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \{y \in X : \|x - y\| \le r\}.$$

For each function $f : X \to R^1$ and each $A \subset X$, put

$$\inf(f) = \inf\{f(z) : z \in X\}$$

and

$$\inf(f; A) = \inf\{f(x) : x \in A\}.$$

For each $x \in X$ and each $B \subset X$, set

$$d(x, B) = \inf\{\|x - y\| : \ y \in B\}.$$

Consider a minimization problem $h(z) \to \min$, $z \in X$, where $h : X \to R^1$ is a continuous bounded from below function. If the space X is infinite-dimensional, then the existence of solutions of the problem is not guaranteed, and in this situation we consider δ-approximate solutions. Namely, $x \in X$ is a δ-approximate solution of the problem $h(z) \to \min$, $z \in X$, where $\delta > 0$, if $h(x) \le \inf(h) + \delta$.

Let $g \in C^2(X; R^1)$. A point $x \in X$ is called a critical point of g if $g'(x) = 0$. Denote by $Cr(g)$ the set of all critical points of g. A real number c is a critical value of g if there exists $x \in Cr(g)$ such that $g(x) = c$. Denote by $Cr(g, +)$ the set of all $x \in Cr(g)$ such that

$$\langle g''(x)u, u \rangle \ge 0 \quad \text{for all } u \in X$$

and by $Cr(g, -)$ the set of all $x \in Cr(g)$ such that

$$\langle g''(x)u, u \rangle \le 0 \quad \text{for all } u \in X.$$

Let $c \in R^1$, $\gamma > 0$ and let $g^{-1}(c) \ne \emptyset$. We assume that g satisfies the following Palais–Smale (P–S) condition [14] on the set $g^{-1}([c - \gamma, c + \gamma])$:

(PS) If $\{x_i\}_{i=1}^{\infty} \subset g^{-1}([c - \gamma, c + \gamma])$ is a bounded sequence (with respect to the norm of X) and if $\lim_{i\to\infty} \|g'(x_i)\| = 0$, then there is a norm convergent subsequence of $\{x_i\}_{i=1}^{\infty}$.

Let a function $f \in C^1(X; R^1)$ be Lipschitzian on all bounded subsets of X and let it satisfy the growth condition

$$\lim_{\|x\|\to\infty} f(x) = \infty \tag{1.1}$$

and let the mapping $f' : X \to X$ be locally Lipschitzian. Clearly, f is bounded from below.