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Transverse bounded solutions to saddle–centers in periodically perturbed ordinary differential equations

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Abstract

Ordinary differential equations are considered consisting of two equations with nonlinear coupling where the linear parts of the two equations have equilibria which are, respectively, a saddle and a center. Perturbation terms are added which correspond to damping and forcing. A reduced equation is obtained from the hyperbolic equation by setting to zero the variable from the center equation with a homoclinic structure. The center equation is resonant in the equilibrium. Melnikov theory is used to obtain a transverse bounded solution of the whole equation. The techniques make use of exponential dichotomies and an averaging procedure.

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1. Introduction

We consider differential equations of the form

$$\dot{x} = f(x, y, \mu, t) = f_0(x, y) + \mu_1 f_1(x, y, \mu, t) + \mu_2 f_2(x, y, \mu, t), \tag{1.1}$$

$$\dot{y} = g(x, y, \mu, t) = g_0(x, y) + \mu_1 g_1(x, y, \mu, t) + \mu_2 g_2(x, y, \mu) \tag{1.2}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$.

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We make the following assumptions about (1.1) and (1.2):

(i) Each f_i, g_i is $\mathcal{C}^4$ in all arguments.
(ii) f_1, f_2 and g_1 are periodic in t with period T.
(iii) $D_2 f_0(x, 0) = 0$.
(iv) The eigenvalues of $D_1 f_0(0, 0)$ lie off the imaginary axis.
(v) The equation $\dot{x} = f_0(x, 0)$ has a homoclinic solution γ.
(vi) $g_0(x, 0) = g_2(x, 0, \mu) = 0$, $D_{21} g_0(0, 0) = 0$ and $D_{22} g_0(0, 0) = 0$.
(vii) The eigenvalues of $D_2 g_0(0, 0)$ lie on the imaginary axis.
(viii) If $\mu_2 \to \lambda(\mu_2)$ is a function such that $\lambda(\mu_2)$ is an eigenvalue of the matrix $D_2 g_0(0, 0) + \mu_2 D_2 g_2(0, 0, 0)$ then $\Re(\lambda'(0)) < 0$.

In the hypothesis (viii), it is sufficient to assume that $\Re(\lambda'(0)) \neq 0$. In other words, (1.2) is weakly hyperbolic with respect to μ_2. This more general assumption requires a little more work since it is necessary to include a nontrivial projection in Lemma 3.2 below.

Consider the equation

$$\dot{x} = f_0(x, 0) + \mu_1 f_1(x, 0, \mu, t) + \mu_2 f_2(x, 0, \mu, t) \tag{1.3}$$

obtained by setting $y = 0$ in (1.1). Eq. (1.3) is called the reduced equation obtained from (1.1) and (1.2). By hypothesis, the equation $\dot{x} = f_0(x, 0)$ has a hyperbolic equilibrium and a homoclinic solution γ. Melnikov theory is used in [11] to obtain a transverse homoclinic solution in the reduced equation (see also [1,9,10]). The problem which naturally arises is showing that a transverse homoclinic solution for the reduced equation is shadowed by a transverse homoclinic solution for the full equation (1.1) and (1.2). This was done in [8] when the center equation

$$\dot{y} = g_0(0, y) + \mu_1 g_1(0, y, \mu, t) + \mu_2 g_2(0, y, \mu) \tag{1.4}$$

is not resonant at $y = 0$. On the other hand, the resonant case is also studied in [8] when a transverse homoclinic solution of the full system is not detected from the reduced equation although with the additional condition $D_{222} g_0(0, 0) = 0$ we develop a Melnikov function containing terms also from the center part. The purpose of the present work is to treat the resonant case without this additional condition and again to detect a transverse homoclinic solution for the full system from a Melnikov function derived from the reduced and center equations. But the situation in this paper is much more delicate than in [8].

The plan of this paper is as follows. In Section 2, we study a concrete system of two coupled second-order ordinary differential equations of the form (1.1) and (1.2). The reduced equation (1.3) now has a Duffing form [17], while the center equation (1.4) is a nonlinear oscillator. We introduce some parameters and then find a concrete numerical relation between them in order to get a transverse homoclinic solution for the perturbed system. Then in Section 3, we generalize the method of Section 2 to arbitrary systems of the form (1.1) and (1.2). In the concluding Section 4, we discuss the use of averaging theory [15] in our approach to improve the results of Section 2.

Finally we note that a related problem is studied also in [5], where a three-dimensional ordinary differential equation is considered with a slowly varying one-dimensional variable. The approach in [5] is more geometrical than ours in this paper.