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Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings

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ABSTRACT

In this paper, we prove strong convergence theorems to a zero of monotone mapping and a fixed point of relatively weak nonexpansive mapping. Moreover, strong convergence theorems to a point which is a fixed point of relatively weak nonexpansive mapping and a solution of a certain variational problem are proved under appropriate conditions.

1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single valued and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see, [6,21]). We note that in a Hilbert space, H, J is the identity operator.

A mapping $A : D(A) \subset E \to E^*$ is said to be *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle \geq 0. \tag{1.1}$$

A is said to be *γ-inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2, \quad \text{for all } x, y \in K. \tag{1.2}$$

If A is γ-inverse strongly monotone, then it is *Lipschitz continuous* with constant $\frac{1}{\gamma}$, i.e., $\|Ax - Ay\| \leq \frac{1}{\gamma}\|x - y\|$, for all $x, y \in D(A)$, and it is called *strongly monotone* if for each $x, y \in D(A)$ there exists $k \in (0, 1)$ such that

$$\langle Ax - Ay, x - y \rangle \geq k\|x - y\|^2. \tag{1.3}$$

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A monotone mapping A is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone mapping. It is known that monotone mapping A is maximal if and only if for $(x, x^*) \in E \times E^*$, $\langle x-y, x^*-y^*\rangle \geq 0$ for every $(y, y^*) \in G(A)$ implies that $x^* \in A(T)$.

The notion of monotone mappings was introduced by Zarantonello [24], Minty [14] and Kacurovskii [9] in Hilbert spaces. Monotonicity conditions in the context of variational methods for nonlinear operator equations were also used by Vainberg and Kacurovskii [22]. This notion has been extended to Banach spaces by several authors (see, e.g., [1,3,7,10,18] and the references contained therein).

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E, \tag{1.4}$$

is studied by Alber [1], Kimimura and Takahashi [10] and Reich [17]. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E.$$

Observe that in a Hilbert space H, (1.4) reduces to $\phi(x, y) = \|x - y\|^2$, for $x, y \in H$.

Let E be a reflexive, strictly convex and smooth Banach space and let K be a nonempty closed and convex subset of E. The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_K : E \to K$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_K x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x), y \in K\}. \tag{1.5}$$

In fact, we have the following result.

Lemma 1.1 (*[1]*). *Let K be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in K$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in K\}$.*

Let K be a nonempty closed convex subset of E, and let T be a mapping from K into itself. We denote by $F(T)$ the set of fixed points of T. A point p in K is said to be *an asymptotic fixed point* of T [3] if K contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from K into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$ and *relatively nonexpansive* (see, e.g., [3,4]) if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in K$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [3–5]. A point p in K is said to be a *strong asymptotic fixed point of* T if K contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from K into itself is called *relatively weak nonexpansive* if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in K$ and $p \in F(T)$.

If E is a smooth, strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}(0) \neq \emptyset$ then it is proved in [12] that $J_r := (J + rA)^{-1}J$, for $r > 0$ is relatively weak nonexpansive. Moreover, if $T : E \to E$ is relatively weak nonexpansive then using the definition of ϕ one can show that $F(T)$ is *closed and convex*.

It is obvious that a relatively nonexpansive mapping is a relatively weak nonexpansive mapping. In fact, for any mapping $T : K \to K$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

Suppose that A is monotone mapping from K into E. The variational inequality problem is formulated as finding

$$\text{a point } u \in K \text{ such that } \langle v - u, Au\rangle \geq 0, \quad \text{for all } v \in K. \tag{1.6}$$

The set of solutions of the variational inequality problem is denoted by $VI(K, A)$. Variational inequalities were initially studied by Stampacchia [11,13] and ever since have been widely studied. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding point $u \in K$ satisfying $VI(K, A)$. If $E = H$, a Hilbert space, one method of solving a point $u \in VI(K, A)$ is the projection algorithm which starts with any point $x_1 = x \in K$ and updates iteratively as x_{n+1} according to the formula

$$x_{n+1} = P_K(x_n - \alpha_n Ax_n), \quad \text{for any } n \geq 1,$$

where P_K is the metric projection from H onto K and $\{\alpha_n\}$ is a sequence of positive real numbers. In the case when A is γ-inverse strongly monotone, Iiduka, Calabashi and Toyoda [7] studied the following iterative scheme (see, e.g., [19],[15]).

$$\begin{cases} x_0 \in K, \text{ chosen arbitrary,} \\ y_n = P_K(x_n - \alpha_n Ax_n) \\ C_n = \{z \in K : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in K : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 1, \text{ for } n \geq 1, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $[0, 2\gamma]$. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $P_{VI(K,A)}(x_0)$, where $P_{VI(K,A)}$ is the metric projection from K onto $VI(K, A)$. In the case that E is a Banach space, Alber [1] proved the following strong convergence theorem to the unique solution of the variational inequality $VI(K, A)$.

Theorem A1 (*[1]*). *Let K be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Suppose an operator A of E into E^* satisfies the following conditions:*

(i) *A is uniformly monotone, that is $\langle x - y, Ax - Ay\rangle \geq \psi(\|x - y\|)$ for all $x, y \in E$, where $\psi(t)$ is a strictly increasing function for all $t \geq 0$ with $\psi(0) = 0$.*
(ii) *$VI(K, A) \neq \emptyset$.*