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Multiple solutions of some nonlinear fourth-order beam equations

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Abstract

Several new existence theorems on three solutions and infinitely many solutions for the following fourth-order beam equation are obtained:

$$u^{(4)} = f(t, u(t)), \quad t \in [0, 1]; \qquad u(0) = u(1) = u''(0) = u''(1) = 0,$$

where $f \in C^1([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$. The Morse theory is employed to discuss this problem.

Keywords: Boundary value problems; Critical points; Critical group; Morse inequality

1. Introduction and main results

It is well known that the following fourth-order two-point boundary value problem (BVP):

$$\begin{cases} u^{(4)} = f(t, u(t)), & t \in [0, 1] \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1}$$

describes the deformation of an elastic beam both of whose ends are simply supported at 0 and 1. In recent years, much attention has been given to BVP (1.1) by a number of authors; see [2,11–14,19] and references therein. Among this literature, most of the authors obtained the existence of positive solutions under the assumption that f is superlinear or sublinear in u by employing the cone expansion or compression fixed point theorem, except [11]. In [11], by using the strongly monotone operator principle and the critical point theory, Li et al. established some sufficient conditions for f to guarantee that the problem has a unique solution, at least one nonzero solution, or infinitely many solutions. In a later paper [9], by combining the critical point theory and the method of subsolutions and supersolutions, some new existence theorems on multiple positive, negative and sign-changing solutions of BVP (1.1) are established. And those theorems therein can deal with the nonlinearity composed of a sublinear function and a superlinear function.

In the present paper, we consider the multiplicity of the solutions to BVP (1.1) and establish a three-solution theorem and an infinitely many-solution theorem by applying the Morse theory. Our methods are different from those of the literature mentioned above. Throughout the paper, we will assume that $f \in C^1([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$. Then the main results can be stated as follows.

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Theorem 1.1. *Assume that the following conditions hold:*

(H_1) $f(t, 0) = 0$ *for* $t \in [0, 1]$*;*

(H_2) *there exist* α *and* $\beta \in \mathbb{R}^1$ *with* $\alpha < \pi^4/2$ *such that*

$$F(t, u) := \int_0^u f(t, v)\mathrm{d}v \leqslant \alpha u^2 + \beta \quad \text{for all } (t, u) \in [0, 1] \times \mathbb{R}^1; \tag{1.2}$$

(H_3) *there exists a natural number* $m \geqslant 1$ *such that*

$$m^4\pi^4 < f'_u(t, 0) < (m + 1)^4\pi^4 \quad \text{for all } t \in [0, 1],$$

where $f'_u(t, u)$ *denotes the first-order derivative of* f *in* u*.*

Then BVP (1.1) has at least three distinct solutions in $C^4[0, 1]$.

Remark 1.2. When dealing with BVP (1.1) using Amann's three-solution theorem [1, Theorem 14.2], we usually assume that $f(t, u)$ is increasing in u and should impose conditions to ensure the existence of subsolutions and supersolutions for BVP (1.1) (see [9] for example). In [10], Henderson and Thompson applied the Leggett–Williams fixed point theorem to obtain a three-solution theorem for some higher even order boundary value problems, where they imposed growth conditions on f on three intervals. Our conditions imposed here are quite different from theirs in the literature.

Theorem 1.3. *Assume that the following conditions hold:*

(H_4) *there exist* $\nu > 2$ *and* $M > 0$ *such that*

$$0 < \nu F(t, u) \leqslant u f(t, u) \quad \text{for all } |u| \geqslant M \text{ and } t \in [0, 1]; \tag{1.3}$$

(H_5) $f(t, u)$ *is odd in* u*, i.e.,* $f(t, -u) = -f(t, u)$ *for all* $(t, u) \in [0, 1] \times \mathbb{R}^1$.

Then BVP (1.1) has infinitely many solutions in $C^4[0, 1]$.

Remark 1.4. In [11], using a symmetric mountain pass lemma [16, Theorem 9.12] due to Rabinowitz, Li et al. obtained an infinitely many-solution result for BVP (1.1) under the following conditions [11, Theorem 3.4]:

(i) $f(t, u) \in C([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$ is odd in u;
(ii) there exist $\mu \in [0, 1/2)$ and $M > 0$ such that $F(t, u) \leqslant \mu u f(t, u)$ for all $|u| \geqslant M$ and $t \in [0, 1]$;
(iii) $\limsup_{u\to 0} f(t, u)/u < \pi^4$, $\liminf_{u\to +\infty} f(t, u)/u = +\infty$ uniformly for $t \in [0, 1]$.

In Theorem 1.3, we replace condition (ii) above by a slightly stronger condition (H_4) and strengthen the differentiability of f. At the same time, we remove condition (iii). In fact, (H_4) implies that $f(t, u)$ is superlinear at $+\infty$ in u; see (3.17) in Section 3. Essentially, the method in [11] is the $\mathbb{Z}_2$-index theory (i.e. genus), while our method is the Morse theory.

We present two simple examples to which Theorems 1.1 and 1.3 can be applied respectively.

Example 1.5. Let

$$f(t, u) = 81u + 4015 \sin u + t \quad \text{for all } (t, u) \in [0, 1] \times \mathbb{R}^1.$$

It is easy to verify that all conditions of Theorem 1.1 are satisfied. So Theorem 1.1 ensures that BVP (1.1) has at least three distinct solutions.

Example 1.6. Let

$$f(t, u) = \pi^4 u + a(t) \arctan u \ln(1 + u^2) + bu^3 \quad \text{for all } (t, u) \in [0, 1] \times \mathbb{R}^1,$$

where $a \in C^1[0, 1]$, $a(t) \geqslant 0$ for $t \in [0, 1]$ and $b > 0$. It is obvious that $f(t, u) \in C^1([0, 1] \times \mathbb{R}^1, \mathbb{R}^1)$ is odd in u. Observe that $F(t, u) > 0$ for all $(t, u) \in [0, 1] \times (0, \infty)$ and

$$\lim_{u\to\infty} \frac{f(t, u)}{u^3} = b > 0.$$

According to Remark 3.1 in [11], we know that (H_4) holds. Theorem 1.3 guarantees that BVP (1.1) has infinitely many solutions. It should be pointed that Theorem 3.4 in [11] cannot be applied to this example.