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Existence of bounded positive solutions of semilinear elliptic equations☆

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Abstract

In this paper, by using fixed point theory, under quite general conditions on the nonlinear term, we obtain an existence result on bounded positive solutions of certain semilinear elliptic equations in $\mathbf{R}^n$, $n \geq 3$.

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Keywords: Positive solution; Semilinear elliptic equations; Fixed point theory

1. Introduction

Let us consider the semilinear elliptic equation

$$\triangle u + f(x,u) + g(|x|)x \cdot \nabla u = 0, \quad x \in U_R \tag{1}$$

where $U_R = \{x \in \mathbf{R}^n : |x| > R\}$ and where $R \geq 0$ and $n \geq 3$, f is locally Hölder continuous in $U_R \times \mathbf{R}$ and $g \in C^1([0,+\infty))$.

A function $u \in C^2(U_R)$ is called a solution of (1) if it satisfies (1) for $x \in U_R$, or a subsolution of (1) if it satisfies $\triangle u + f(x, u(x)) + g(|x|)x \cdot \nabla u \geq 0$ for $x \in U_R$, or a supersolution of (1) if it satisfies $\triangle u + f(x, u(x)) + g(|x|)x \cdot \nabla u \leq 0$ for $x \in U_R$.

In our discussion, the following lemma is needed because we use the technique of super/subsolutions. Set $S_R = \{x \in \mathbf{R}^n : |x| = R\}$.

Lemma 1.1 (*See [2]*). *Assume that f is locally Hölder continuous in $U_R \times \mathbf{R}$ and $g \in C^1([0,+\infty))$. If for some $R \geq 0$, there exists a positive subsolution $\omega(x)$ and a positive supersolution $\nu(x)$ of (1) in U_R such that $\omega(x) \leq \nu(x)$ for $x \in U_R \cup S_R$, then (1) has a solution u(x) in U_R such that $\omega(x) \leq u(x) \leq \nu(x)$ for $x \in U_R \cup S_R$ and $u(x) = \nu(x)$ for $x \in S_R$.*

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In recent years, several authors have investigated the existence of positive solutions of (1), and obtained many good results (see [1–4,10,11,13–15]). But in these results, the conditions that $f(x,0) \geq 0$, k is large, where k is the radius of the boundary, there exist $a \in C([0,\infty),[0,\infty))$ and $b \in C^1([0,\infty),[0,\infty))$ with $a(|x|)b(t) \geq |f(x,t)|$ such that

$$\int_0^\infty s[a(s) + |g(s)|]\mathrm{d}s < \infty \tag{2}$$

(see [13,15]) or there exist some $c > 0$ and a dominating function $h(t,s)$ which is non-decreasing on s with $h(|x|, u(x)) \geq |f(x, u(x))|$ such that

$$\int_0^\infty sh(s,c)\mathrm{d}s < \infty \tag{3}$$

(see [13]) were assumed.

When the nonlinearity $f(x, u(x))$ is radial, i.e., $f(x,u(x)) = f(|x|, u(x))$, a positive solution of (1) comes right from the ODE

$$u''(t) + l(u(t))u'(t) + k(t, u(t)) = 0, \quad t \geq 0 \tag{4}$$

where $k \in C([0,\infty) \times \mathbf{R}, \mathbf{R})$, $l \in C(\mathbf{R}, \mathbf{R})$.

For the existence of bounded positive global solutions of (4), the reader is referred to the papers [1–13,15].

For comparison with previous works, we give the following two examples.

Example 1.1. Consider the following equation:

$$\triangle u + \frac{u \sin|x|}{A(1+|x|)^2} + \frac{10\cos|x| x \cdot \nabla u}{B(1+|x|)^2} = 0, \quad x \in \mathbf{R}^n, |x| \geq 1 \text{ and } n \geq 3 \tag{5}$$

where A and B are constants with $|A|, |B| \geq 100$.

Remark 1.2. In this example, although the condition $f(x,0) \geq 0$ is satisfied, the previous results cannot be applied, since (2) and (3) do not hold.

Example 1.3. Consider the following equation:

$$\triangle u + \frac{u \sin|x|}{A(1+|x|)^\alpha} + \frac{\cos|x| x \cdot \nabla u}{B(1+|x|)^\alpha} + \frac{\xi \cdot x}{C(1+|\xi||x|)(1+|x|)^3} = 0,$$
$$x \in \mathbf{R}^n, |x| \geq 1 \text{ and } n \geq 3 \tag{6}$$

where $\alpha > 2$, A, B and C are constants with $|A|, |B||C| > 100$ and $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ where ξ_1 and ξ_1 are constants with $\xi_1 \times \xi_2 \neq 0$.

Remark 1.4. In this example, the previous results cannot be applied, since $f(x,0)$ is sign-changing.

Therefore, it is of significance to investigate the existence of bounded positive solutions of (1) under the condition that $f(x,0)$ is sign-changing or the condition that (2) or (3) is relaxed. Hence, it is much more significant to investigate the existence of bounded positive solutions of (1) under the conditions where $f(x,0)$ is sign-changing and (2) and (3) are relaxed. The latter is also our aim in this paper (see Examples 1.1, 1.3 and 3.2).

In Section 3, by our main result, we shall prove that (5) and (6) have bounded positive solutions and give a new example.

In Section 2, we consider the existence of bounded positive global solutions of the equation $v''(t) + f(t, v(t), v'(t)) = 0$ and obtain a sufficient condition for the existence of bounded positive global solutions of this equation and its corollary. For several cases, it is obvious that this result improves the results of [1–15] (see Examples 1.1, 1.3 and 3.2).

In the last section, under the conditions that $f(x,0)$ is sign-changing and (2) and (3) are relaxed, we apply the results of the second section and the technique of super/subsolutions to the quasilinear second-order elliptic equation (1) and prove that there exists a bounded positive solution to (1). This is our main result in this paper. For several cases, the result improve the results of [11–15] (see Examples 1.1, 1.3 and 3.2).