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Stability of affine coalescence hidden variable fractal interpolation functions[☆]

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Abstract

The stability of an affine coalescence hidden variable fractal interpolation function is proved in a general set up in the present work, by establishing that any small perturbation in the generalized interpolation data leads to a small perturbation in the corresponding affine coalescence hidden variable fractal interpolation function.

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1. Introduction

The term ‘fractals’ was coined by Mandelbrot [1] in order to describe objects that did not easily fit into the standard Euclidean geometric settings. Fractal curves have been applied extensively in the natural sciences [1–3] and engineering [4,5] in the past 30 years. Iterated function systems (IFSs) were introduced by Barnsley [6,7] to approximate these curves by fractal interpolation functions (FIFs). FIFs are generally self-affine in nature, and the Hausdorff–Besicovitch dimensions of their graphs are non-integers. To approximate non-self-affine patterns, hidden variable FIFs (HFIFs) are constructed in [7–9] by projection of vector valued FIFs from suitable generalized interpolation data.

However, in practical applications of FIFs, the interpolation data might be generated simultaneously from self-affine and non-self-affine functions. To construct self-affine and non-self-affine objects simultaneously from an IFS, the concept of *constrained free variables* was introduced [10] to generate hidden variable bivariate fractal interpolation surfaces. The construction of coalescence hidden variable fractal interpolation functions (CHFIFs) [11, 12] indicates that they are ideal tools to approximate self-affine or non-self-affine objects simultaneously, depending

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on the parameters of the defined IFS. In the present paper, stability results of the affine CHFIF are found in a more general set-up than that of Feng and Xie [14].

In Section 2, a brief introduction to the CHFIF is given. The smoothness results and some other auxiliary results for CHFIFs are derived in Section 3. Our main stability result from a pool of generalized interpolation data is derived by using the stability results of co-ordinate wise perturbations in Section 4.

2. Coalescence hidden variable FIF

Let the real interpolation data be $\{(x_i, y_i) : i = 0, 1, 2, \ldots, N\}$, where $-\infty < x_0 < x_1 < \cdots < x_N < +\infty$. To construct a CHFIF $f_1 : [x_0, x_N] \to \mathbb{R}$ such that $f_1(x_i) = y_i$ for all $i = 0, 1, 2, \ldots N$, extend the given data to generalized interpolation data as $\{(x_i, y_i, z_i) \in \mathbb{R}^3 | i = 0, 1, 2, \ldots, N\}$, where $z_i, i = 0, 1, 2, \ldots, N$ are real parameters. The following notations are used in the construction of a CHFIF: $I = [x_0, x_N]$, $I_i = [x_{i-1}, x_i]$, $\tilde{g}_1 = \text{Min}_i\, y_i$, $\tilde{g}_2 = \text{Max}_i\, y_i$, $\tilde{h}_1 = \text{Min}_i\, z_i$, $\tilde{h}_2 = \text{Max}_i\, z_i$, $K = I \times D$, where $D = J_1 \times J_2$, J_1 and J_2 are suitable compact sets in $\mathbb{R}$ such that $[\tilde{g}_1, \tilde{g}_2] \subseteq J_1$ and $[\tilde{h}_1, \tilde{h}_2] \subseteq J_2$. Let $L_i : I \longrightarrow I_i$ be a contractive homeomorphism and $F_i : K \longrightarrow D$ be a continuous vector valued function such that

$$\begin{cases} L_i(x_0) = x_{i-1}, \qquad L_i(x_N) = x_i, \\ F_i(x_0, y_0, z_0) = (y_{i-1}, z_{i-1}), \qquad F_i(x_N, y_N, z_N) = (y_i, z_i), \end{cases} \tag{2.1}$$

and

$$\begin{cases} d(F_i(x, y, z), F_i(x^*, y, z)) \leq c|x - x^*|, \\ d(F_i(x, y, z), F_i(x, y^*, z^*)) \leq s\, d_E((y, z), (y^*, z^*)), \end{cases} \tag{2.2}$$

for all $i = 1, 2, \ldots, N$, c and s are positive constants with $0 \leq s < 1$, $(x, y, z), (x^*, y, z), (x, y^*, z^*) \in K$, d is the sup. metric on K and d_E is the Euclidean metric on $\mathbb{R}^2$. To define a CHFIF, we choose, $L_i(x) = a_i x + b_i$ and

$$F_i(x, y, z) = A_i(y, z)^{\mathrm{T}} + (p_i(x), q_i(x))^{\mathrm{T}}, \tag{2.3}$$

where A_i is an upper triangular matrix as $\begin{pmatrix} \alpha_i & \beta_i \\ 0 & \gamma_i \end{pmatrix}$ and $p_i(x)$, $q_i(x)$ are continuous functions having at least two unknowns. We choose α_i as a free variable with $|\alpha_i| < 1$ and β_i as a *constrained free variable* with respect to γ_i such that $|\beta_i| + |\gamma_i| < 1$ for $i = 1, 2, \ldots, N$. The generalized IFS for the construction of a CHFIF is now defined as

$$\{\mathbb{R}^3; \omega_i(x, y, z) = (L_i(x), F_i(x, y, z)), i = 1, 2, \ldots, N\}. \tag{2.4}$$

Barnsley et al. [8] proved that the IFS defined in (2.4) associated with the data $\{(x_i, y_i, z_i), i = 0, 1, \ldots, N\}$, is hyperbolic with respect to a metric d^* on $\mathbb{R}^3$, where d^* is equivalent to the Euclidean metric. In particular, there exists a unique nonempty compact set $G^* \subseteq \mathbb{R}^3$ such that $G^* = \bigcup_{i=1}^{N} \omega_i(G^*)$. The following proposition from [8] gives the existence of a unique vector valued function f that interpolates the generalized interpolation data and also establishes that the graph of f equals the attractor G^* of the generalized IFS.

Proposition 2.1. *The attractor G^* of the IFS defined in (2.4) is the graph of the continuous vector valued function $f : I \longrightarrow D$ such that $f(x_i) = (y_i, z_i)$ for all $i = 1, 2, \ldots, N$ i.e. $G = \{(x, y, z) : x \in I$ and $f(x) = (y(x), z(x))\}$.*

Let the vector valued function $f : I \to D$ in Proposition 2.1 be written as $f(x) = (f_1(x), f_2(x))$. Let $\{(x, f_1(x)) : x \in I\}$ be the projection of the attractor G^* on $\mathbb{R}^2$. Then, the function $f_1(x)$ is called the *coalescence hidden variable FIF*(CHFIF) for the given interpolation data $\{(x_i, y_i) | i = 0, 1, \ldots, N\}$. Also, the function $f_2(x)$ in the projection $\{(x, f_2(x)) : x \in I\}$ by taking 1st and 3rd co-ordinates of G^* is a self-affine fractal function that interpolates the data $\{(x_i, z_i) | i = 0, 1, \ldots, N\}$. Since the projection of the attractor is not always a union of affine transformations of itself, a CHFIF is generally non-self-affine in nature. If we choose $y_i = z_i$, $\alpha_i + \beta_i = \gamma_i$ and $p_i(x) = q_i(x)$, the CHFIF $f_1(x)$ coincides with the self-affine FIF $f_2(x)$ for the same interpolation data.

Proposition 2.1 gives that the graph of the vector valued function $f(x) = (f_1(x), f_2(x))$ is the attractor of the IFS $\{\mathbb{R}^3; \omega_i(x, y, z), i = 1, 2, \ldots, N\}$ if and only if the fixed point f satisfies

$$f(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))), \tag{2.5}$$