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Nontrivial Fučík spectrum of one non-selfadjoint operator

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Abstract

We provide complete analytical description of the Fučík spectrum of the non-selfadjoint operator corresponding to four-point boundary value problem. The non-selfadjointness results in a surprising structure of the Fučík spectrum with new interesting patterns.

MSC: 34B10; 34B15; 34L05

Keywords: Fučík spectrum; Asymmetric nonlinearities; Multi-point boundary value problem; Non-selfadjoint operator

1. Introduction

In this paper we study the structure of the Fučík spectrum of the linear second order operator with four-point boundary conditions

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \qquad u(\pi) = u(\eta), & \xi \in (0,\pi),\ \eta \in (0,\pi), \end{cases} \tag{1}$$

that is, we are looking for such pairs $(\alpha, \beta) \in \mathbb{R}^2$, for which the problem (1) has a nontrivial solution. We usually denote this set by

$$\Sigma(L) := \left\{ (\alpha, \beta) \in \mathbb{R}^2 : Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution} \right\},$$

where L is a linear (scalar differential) operator and $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$.

The Fučík spectrum and its importance were discovered by Fučík [5] and Dancer [2] while studying the solvability of sublinear boundary value problems $Lu = f(u)$, where L is some linear operator and f is a nonlinearity having linear growth in infinity: $\lim_{s\to+\infty} \frac{f(s)}{s} = \alpha$, $\lim_{s\to-\infty} \frac{f(s)}{s} = \beta$. Fučík spectra of various linear operators have been studied by many authors by both analytical and numerical tools (see, e.g., [1,3,7,9,11–14]). However, the majority of the known results concerns *selfadjoint* operators. This property brings a big advantage since it enables us to use

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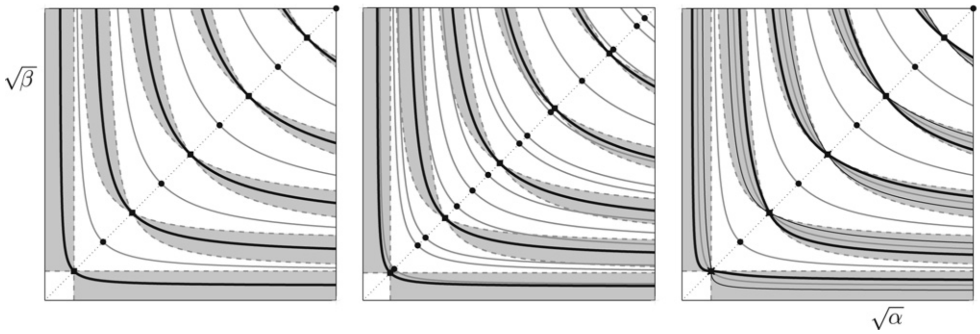

Fig. 1. Trivial patterns of the Fučík spectrum of (1): simple curves with no intersections (left and middle), no intersections away from the diagonal (right).

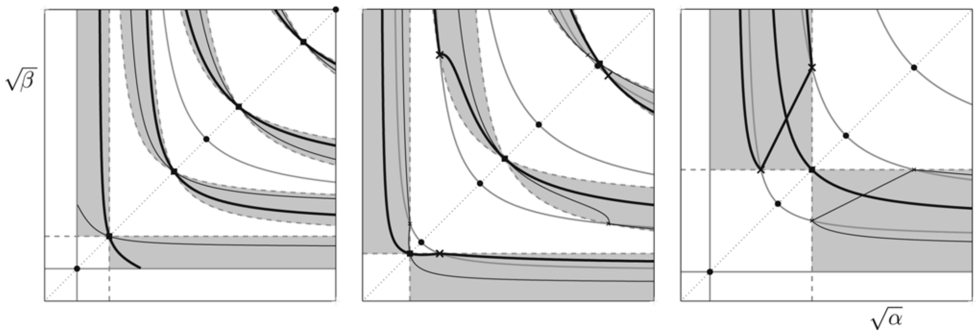

Fig. 2. Pathological patterns of the Fučík spectrum of (1): cut of the first nontrivial branch (left), loss of monotonicity and smoothness of branches (middle) and trident pattern (right).

variational techniques, but it also influences essentially the structure of the Fučík spectrum. Main features of the Fučík spectrum $\Sigma(L)$ of a general linear selfadjoint operator $L : L^2(\Omega) \to L^2(\Omega)$, Ω being open and bounded in $\mathbb{R}^N$, were described by Ben-Naoum, Fabry and Smets in [1] and can be summarized as follows:

(1) $\Sigma(L)$ is symmetric with respect to the diagonal $\alpha = \beta$ and $(\lambda, \lambda) \in \Sigma(L)$ for any real eigenvalue λ of L.
(2) Under certain nondegeneracy conditions (see [1]), the Fučík spectrum $\Sigma(L)$ of a selfadjoint operator L consists locally of a finite number of curves crossing at (λ, λ).
(3) Each of these curves can be associated with a critical point of the functional $x \mapsto \langle |x|, x\rangle$ restricted to the unit sphere in $\mathrm{Ker}(L - \lambda I)$. These extrema can be grouped in pairs and they correspond to Fučík curves, which are symmetric with respect to the diagonal $\alpha = \beta$. The number of these pairs can be higher than the dimension of $\mathrm{Ker}(L - \lambda I)$ (i.e., higher than the number of eigenfunctions corresponding to λ).
(4) These curves can be continued up to the boundary of $J \times J$, where J is an interval which has no intersection with the essential spectrum of L.
(5) Under an additional nondegeneracy condition (see [1]), any Fučík curve can be described by a monotone differentiable function $\alpha \mapsto \beta(\alpha)$ with

$$\beta'(\alpha) = -\frac{\|u_0^+\|^2}{\|u_0^-\|^2} < 0,$$

where u_0 is a solution of $Lu_0 = \alpha u_0^+ - \beta u_0^-$.

Notice that the first property is valid for any (even non-selfadjoint) operator.

If we consider an ordinary second order operator $Lu = -u''$ with standard boundary conditions (Dirichlet, Neumann or periodic boundary conditions), the structure of the Fučík spectrum is well-known and very simple: it