Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 69 (2008) 2984–2995
Nonlinear Analysis
www.elsevier.com/locate/na

Necessary and sufficient conditions for the existence of solution to three-point BVP

G. Bognár[a], J. Čepička[b], P. Drábek[b], P. Nečesal[b,*], E. Rozgonyi[a]

[a] *Institute of Mathematics, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary*
[b] *University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic*

Received 26 May 2007; accepted 30 August 2007

Abstract

We study the existence of a solution of the three-point boundary value problem

$$x'' + \lambda x + g(x) = h, \quad x(0) = 0,\ x(\eta) = x(\pi). \tag{P}$$

We give sufficient and necessary conditions on g and h in order for (P) to have a real solution. Our conditions are of Landesman–Lazer type and depend on the real values of λ and η.

MSC: 34B10; 34B15

Keywords: Resonance at higher eigenvalues; Landesman–Lazer type condition; Leray–Schauder degree; Three-point boundary value problem

1. Introduction

In this paper we deal with the three-point boundary value problem

$$x'' + \lambda x + g(x) = h, \quad x(0) = 0,\ x(\eta) = x(\pi), \tag{1}$$

where $h \in L^1(0, \pi)$ is a given real function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function and $\lambda \in \mathbb{R}$ and $\eta \in [0, \pi)$ are parameters. We provide necessary and sufficient conditions for the existence of a real solution of (1) depending on the values of parameters λ and η. In particular, we focus on those pairs (η, λ) for which the linear homogeneous problem

$$x'' + \lambda x = 0, \quad x(0) = 0,\ x(\eta) = x(\pi), \tag{2}$$

has a nonzero real solution.

* Corresponding author.
E-mail addresses: matvbg@uni-miskolc.hu (G. Bognár), cepicka@kma.zcu.cz (J. Čepička), pdrabek@kma.zcu.cz (P. Drábek), pnecesal@kma.zcu.cz (P. Nečesal), matre@uni-miskolc.hu (E. Rozgonyi).

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.08.072

Let us denote

$$\sigma := \{(\eta, \lambda) \in [0, \pi) \times \mathbb{R} : (2) \text{ has a nonzero real solution}\}.$$

A real function $x \in W^{2,1}(0, \pi)$ is called a *solution* of (1) if the equation is satisfied almost everywhere in $(0, \pi)$ and the boundary conditions are satisfied pointwise. In what follows all constants are *real* and all functions are *real-valued*.

The main results can be summarized as follows:

(1) *If* $(\eta, \lambda) \notin \sigma$ *then for any* $h \in L^1(0, \pi)$ *there exists at least one solution* $x \in W^{2,1}(0, \pi)$ *of* (1) (see Remark 5).

If $(\eta, \lambda) \in \sigma$, we have to first characterize the set of all $h \in L^1(0, \pi)$ for which the linear nonhomogeneous problem

$$x'' + \lambda x = h, \quad x(0) = 0, \; x(\eta) = x(\pi) \tag{3}$$

has a solution. Let $\psi = \psi(t)$ be such that (3) has a solution if and only if

$$\int_0^\pi h(t)\psi(t)\,\mathrm{d}t = 0$$

(see Section 2 for explicit formula of ψ). Let $(\eta, \lambda) \in \sigma$ and let us denote by $\varphi = \varphi(t)$ a nontrivial solution of (2) and normalize it by

$$\int_0^\pi \varphi(t)\psi(t)\,\mathrm{d}t \ge 0$$

(see Section 2 for explicit formula of φ). We assume that the limits

$$g(\pm\infty) = \lim_{s \to \pm\infty} g(s)$$

exist and that $g(+\infty) < g(-\infty)$.

(2) *If* $(\eta, \lambda) \in \sigma$ *and*

$$g(+\infty)\int_{\varphi>0} \psi\,\mathrm{d}t + g(-\infty)\int_{\varphi<0} \psi\,\mathrm{d}t < \int_0^\pi h(t)\psi(t)\,\mathrm{d}t < g(-\infty)\int_{\varphi>0} \psi\,\mathrm{d}t + g(+\infty)\int_{\varphi<0} \psi\,\mathrm{d}t \tag{4}$$

then (1) *has a solution* (see Theorem 4).

Moreover, we assume that for all $s \in \mathbb{R}$,

$$g(+\infty) < g(s) < g(-\infty).$$

(3) *If* $(\eta, \lambda) \in \sigma$ *and* (1) *has a solution then necessarily*

$$g(+\infty)\int_{\psi>0} \psi\,\mathrm{d}t + g(-\infty)\int_{\psi<0} \psi\,\mathrm{d}t < \int_0^\pi h(t)\psi(t)\,\mathrm{d}t < g(-\infty)\int_{\psi>0} \psi\,\mathrm{d}t + g(+\infty)\int_{\psi<0} \psi\,\mathrm{d}t \tag{5}$$

(see Theorem 6).

If $g(+\infty) < 0 < g(-\infty)$, we can compare conditions (4) and (5). Namely, we get

$$g(-\infty)\int_{\varphi>0} \psi\,\mathrm{d}t + g(+\infty)\int_{\varphi<0} \psi\,\mathrm{d}t \le g(-\infty)\int_{\psi>0} \psi\,\mathrm{d}t + g(+\infty)\int_{\psi<0} \psi\,\mathrm{d}t \tag{6}$$

and

$$g(+\infty)\int_{\varphi>0} \psi\,\mathrm{d}t + g(-\infty)\int_{\varphi<0} \psi\,\mathrm{d}t \ge g(+\infty)\int_{\psi>0} \psi\,\mathrm{d}t + g(-\infty)\int_{\psi<0} \psi\,\mathrm{d}t. \tag{7}$$