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ScienceDirect
Nonlinear Analysis 69 (2008) 3005–3015
Nonlinear Analysis
www.elsevier.com/locate/na

The first Darboux problem for wave equations with a nonlinear positive source term

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Received 7 June 2007; accepted 29 August 2007

Abstract

We consider the first Darboux problem for nonlinear wave equations with positive power nonlinearity source term. Depending on the power of nonlinearity we investigate the problem on a global existence and blow-up of solutions of the first Darboux problem. The question of local solvability of the problem is also considered.

Keywords: Nonlinear wave equations; The first Darboux problem; Global and local solvability; Blow-up

1. Statement of a problem

In the plane of independent variables x and t consider nonlinear wave equation of the following form

$$Lu := u_{tt} - u_{xx} - |u|^{\alpha} = f(x,t), \tag{1}$$

where $\alpha > 0$ is a real constant, f – given, and u – unknown real functions.

Denote by $D_T := \{(x,t) : -kt < x < t,\ 0 < t < T,\ 0 \le k := \text{const} < 1\}$, $T \le \infty$ triangular domain, located within characteristic angle $\{(x,t) \in \mathbb{R}^2 : t > |x|\}$ and bounded by characteristic segment $\gamma_{1,T} : x = t,\ 0 \le t \le T$, and by segments $\gamma_{2,T} : x = -kt,\ 0 \le t \le T,\ \gamma_{3,T} : t = T,\ -kT \le x \le T$.

For Eq. (1) in domain D_T consider the first Darboux problem on determination of solution $u(x,t)$ by boundary conditions [1, p. 228]:

$$u|_{\gamma_{i,T}} = 0, \quad i = 1, 2. \tag{2}$$

Many works are devoted to the questions of existence and nonexistence of global solutions of nonlinear hyperbolic equations for different problems (such as initial, mixed and nonlocal problems, including periodical) [2–11]. In the linear case, i.e. for $\alpha = 0$, problem (1) and (2) is posed correctly and we have global solvability in the corresponding functional spaces [1,12].

We show that for a certain assumption on the power of nonlinearity α problem (1) and (2) in some cases is globally solvable, while in other cases it has no global solution, though, as it will be shown below, this problem is locally solvable.

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0362-546X/$ - see front matter
doi:10.1016/j.na.2007.08.068

Definition 1. Let $f \in C(\overline{D}_T)$. Function u is called a strong generalized solution of problem (1) and (2) of class C in domain D_T, if $u \in C(\overline{D}_T)$ and there exists such a sequence of functions $u_n \in \overset{o}{C}{}^2(\overline{D}_T, \Gamma_T)$, that $u_n \to u$ and $Lu_n \to f$ in space $C(\overline{D}_T)$ for $n \to \infty$, where $\overset{o}{C}{}^2(\overline{D}_T, \Gamma_T) := \{u \in C^2(\overline{D}_T) : u|_{\Gamma_T} = 0\}$, $\Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}$.

Remark 1. It is clear that if a classical solution of problem (1) and (2) belongs to space $\overset{o}{C}{}^2(\overline{D}_T, \Gamma_T)$, then it is a strong generalized solution of this problem of class C in domain D_T. In turn, if a strong generalized solution of problem (1) and (2) of class C in domain D_T belongs to space $C^2(\overline{D}_T)$, then it also is a classical solution of the problem.

Definition 2. Let $f \in C(\overline{D}_\infty)$. We say that problem (1) and (2) is globally solvable in the class C, if for any finite $T > 0$ the problem has a strong generalized solution of the class C in domain D_T.

2. A priori estimate of the solution of problem (1) and (2)

Lemma 1. *Let $0 < \alpha \le 1$. Then for the strong generalized solution of problem (1) and (2) of class C in domain D_T it is valid for the following a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)} + c_2 \tag{3}$$

with positive constants $c_i(T, \alpha)$, $i = 1, 2$, not dependent on u and f.

Proof. Let u be a strong generalized solution of problem (1) and (2) of class C in domain D_T. Then due to Definition 1 there exists the sequence of functions $u_n \in \overset{o}{C}{}^2(\overline{D}_T, \Gamma_T)$, such that

$$\lim_{n\to\infty} \|u_n - u\|_{C(\overline{D}_T)} = 0, \qquad \lim_{n\to\infty} \|Lu_n - f\|_{C(\overline{D}_T)} = 0, \tag{4}$$

and therefore

$$\lim_{n\to\infty} \||u_n|^\alpha - |u|^\alpha\|_{C(\overline{D}_T)} = 0. \tag{5}$$

Consider the function $u_n \in \overset{o}{C}{}^2(\overline{D}_T, \Gamma_T)$, as a solution of the following problem

$$Lu_n = f_n, \tag{6}$$

$$u_n|_{\Gamma_T} = 0, \qquad \Gamma_T := \gamma_{1,T} \cup \gamma_{2,T}. \tag{7}$$

Here

$$f_n := Lu_n. \tag{8}$$

Multiplying both sides of the equality (6) by $\frac{\partial u_n}{\partial t}$ and integrating the received in domain $D_\tau := \{(x, t) \in D_T : 0 < t < \tau\}$, $0 < \tau \le T$ we have

$$\frac{1}{2}\int_{D_\tau} \frac{\partial}{\partial t}\left(\frac{\partial u_n}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}t - \int_{D_\tau} \frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial t}\mathrm{d}x\mathrm{d}t - \frac{1}{\alpha+1}\int_{D_\tau} \frac{\partial}{\partial t}(|u_n|^\alpha u_n)\mathrm{d}x\mathrm{d}t = \int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\mathrm{d}x\mathrm{d}t.$$

Assume that $\Omega_\tau := \overline{D}_\infty \cap \{t = \tau\}$, $0 < \tau \le T$. Then by virtue of (7), integrating by parts the left side of the last equality, in the case when $k \in (0, 1)$ we have

$$\begin{aligned}\int_{D_\tau} f_n \frac{\partial u_n}{\partial t}\mathrm{d}x\mathrm{d}t &= \int_{\Gamma_\tau} \frac{1}{2\nu_t}\left[\left(\frac{\partial u_n}{\partial x}\nu_t - \frac{\partial u_n}{\partial t}\nu_x\right)^2 + \left(\frac{\partial u_n}{\partial t}\right)^2(\nu_t^2 - \nu_x^2)\right]\mathrm{d}s \\ &\quad + \frac{1}{2}\int_{\Omega_\tau}\left[\left(\frac{\partial u_n}{\partial t}\right)^2 + \left(\frac{\partial u_n}{\partial x}\right)^2\right]\mathrm{d}x - \frac{1}{\alpha+1}\int_{\Omega_\tau} |u_n|^\alpha u_n \mathrm{d}x,\end{aligned} \tag{9}$$

where $\nu := (\nu_x, \nu_t)$ is a unit vector of outer normal to ∂D_τ and $\Gamma_\tau := \Gamma_T \cap \{t \le \tau\}$.