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Sharp estimates of the convergence rate for a semilinear parabolic equation with supercritical nonlinearity

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Abstract

We study the behavior of solutions of the Cauchy problem for a semilinear parabolic equation with supercritical nonlinearity. It is known that if two solutions are initially close enough near the spatial infinity, then these solutions approach each other. In this paper, we give its sharp convergence rate for a class of initial data. We also derive a universal lower bound of the convergence rate which implies the optimality of the result. Proofs are given by a comparison method based on matched asymptotics expansion.

MSC: 35K15; 35B35; 35B40

Keywords: Cauchy problem; Semilinear heat equation; Stationary solution; Convergence; Supercritical exponent

1. Introduction

In this paper, we investigate the behavior of solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, Δ is the Laplace operator with respect to x, $p > 1$, and $u_0 \not\equiv 0$ is a given continuous function on $\mathbb{R}^N$ that decays to zero as $|x| \to \infty$. The problem (1.1) has been studied in many papers, since Fujita studied the blow-up problem [6]. Among them, the stability problem of stationary solutions is one of the most important problems and we study the problem (1.1) along this line.

It is known that there exist critical exponents p that govern the structure of solutions. The exponent

$$p_S = \begin{cases} \dfrac{N+2}{N-2} & N > 2, \\ \infty & N \le 2, \end{cases}$$

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is well-known as the Sobolev exponent that is critical for the existence of positive stationary solution of (1.1). Namely, there exists a classical positive radial solution φ of

$$\Delta\varphi + \varphi^p = 0, \quad x \in \mathbb{R}^N,$$

if and only if $p \geq p_S$ [1,2,8]. We denote the solution by $\varphi = \varphi_\alpha(r)$, $r = |x|$, $\alpha > 0$, where $\varphi_\alpha(0) = \alpha$. Then $\varphi_\alpha(r)$ satisfies the initial value problem

$$\begin{cases} \varphi_{\alpha,rr} + \dfrac{N-1}{r}\varphi_{\alpha,r} + \varphi_\alpha^p = 0, \\ \varphi_\alpha(0) = \alpha, \qquad \varphi_{\alpha,r}(0) = 0. \end{cases}$$

For each $\alpha > 0$, the solution φ_α is strictly decreasing in $|x|$ and satisfies $\varphi_\alpha \to 0$ as $|x| \to \infty$. We extend the solution by setting $\varphi_\alpha = -\varphi_{-\alpha}$ for $\alpha < 0$ and $\varphi_0 = 0$. Then the set $\{\varphi_\alpha; \alpha \in \mathbb{R}\}$ forms a one-parameter family of radial stationary solutions.

The exponent

$$p_c = \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & N > 10, \\ \infty & N \leq 10, \end{cases}$$

is another important exponent which appeared first in [11]. It is known that for $p_S \leq p < p_c$, any pair of positive stationary solutions intersects each other. For $p \geq p_c$, Wang [15] showed that the family of stationary solutions forms a simply ordered set, that is, φ_α is strictly increasing in α for each x. We call it the ordering property of $\{\varphi_\alpha\}$. Moreover, φ_α satisfies

$$\lim_{\alpha\to 0} \varphi_\alpha(|x|) = 0, \quad \lim_{\alpha\to\infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|),$$

for each x, where $\varphi_\infty(|x|)$ is a singular stationary solution given by

$$\varphi_\infty(|x|) = L|x|^{-m}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

with

$$m = \frac{2}{p-1}, \quad L = \{m(N-2-m)\}^{1/(p-1)}.$$

It was also shown in [9] that each positive stationary solution has the expansion

$$\varphi_\alpha(|x|) = \begin{cases} L|x|^{-m} - a_\alpha |x|^{-m-\lambda_1} + \text{h.o.t.} & p > p_c, \\ L|x|^{-m} - a_\alpha |x|^{-m-\lambda_1} \log|x| + \text{h.o.t.} & p = p_c, \end{cases}$$

as $|x| \to \infty$, where λ_1 is a positive constant given by

$$\lambda_1 = \lambda_1(N, p) := \frac{N-2-2m - \sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

and $a_\alpha = a(\alpha)$ is a positive number that is monotone decreasing in α. Note that λ_1 is a smaller root of the quadratic equation

$$h(\lambda) := \lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0.$$

We define by

$$\lambda_2 = \lambda_2(N, p) := \frac{N-2-2m + \sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

a larger root of the quadratic equation. Concerning the stability problem, Gui, Ni and Wang [9,10] showed that any regular positive radial stationary solution is unstable in any reasonable sense if $p_S < p < p_c$ and "weakly asymptotically stable" in a weighted L^∞ norm if $p \geq p_c$. For $p > p_c$, Poláčik and Yanagida [13,14] improved the above results and proved that the solutions approach a set of stationary solutions for a wide class of the initial data.