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A nonlinear equation in Banach spaces and applications to the well-posedness of Cauchy problems

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Abstract

We analyze a nonlinear equation in Banach spaces, with the nonlinearity composed of multiple terms of different degrees. We prove a theorem regarding the existence of solutions for such equations. Moreover, we show how this result may be applied to obtain the well-posedness of various parabolic initial value problems.

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1. Introduction

Let X be a Banach space with norm $\|\cdot\|$. Given $y \in X$, and the operators $T : X \to X$, and $S_i : X \to X$, $1 \le i \le m$, consider the equation

$$x = y + \sum_{i=1}^{m} S_i(x) + T(x). \tag{1}$$

We prove that under suitable conditions Eq. (1) is locally uniquely solvable. Moreover, we show that this result may be used to obtain the well-posedness of various initial value problems for parabolic equations in suitable spaces. More specifically, in Section 3 we derive bounds which allow one to get well-posedness for semilinear parabolic systems and for convection–diffusion equations. Moreover, we also use some known estimates to study other equations, such as the semilinear heat equations and the Navier–Stokes equations.

When necessary, we will denote the norm in a given Banach space X by $\|\cdot\|_X$. For $r > 0$, we write by $B_r := \{x \in X \; ; \; \|x\| \le r\}$. For each Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the rearrangement f^* and the average function f^{**} are, respectively, defined by

$$f^*(t) = \inf_{s>0} \left\{ \mu\left(\{x \in \mathbb{R}^n; |f(x)| > s\}\right) \le t \right\}, \quad t > 0,$$

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$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0,$$

where μ is the $\mathbb{R}^n$-Lebesgue measure. The Lorentz space $L^{(p,q)}(\mathbb{R}^n) \equiv L^{(p,q)}$ consists of all measurable functions f such that $\|f\|_{(p,q)} < \infty$, where

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q \mathrm{d}t/t\right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty,\ 1 \le q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^{**}(t), & \text{if } 1 < p \le \infty,\ q = \infty. \end{cases}$$

We observe that $L^p(\mathbb{R}^n) = L^{(p,p)}(\mathbb{R}^n)$ and $L^{(p,q_1)}(\mathbb{R}^n) \subset L^{(p,q_2)}(\mathbb{R}^n)$, for $q_1 \le q_2$, with a continuous injection. The Lorentz spaces can be constructed via the real interpolation

$$L^{(p,q)}(\mathbb{R}^n) = (L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1-\frac{1}{p},q}, \quad 1 < p < \infty.$$

Moreover, they have the interpolation property

$$(L^{(p_0,q_0)}(\mathbb{R}^n), L^{(p_1,q_1)}(\mathbb{R}^n))_{\theta,q} = L^{(p,q)}(\mathbb{R}^n),$$

provided $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1 \le q_0, q_1, q \le \infty$. The space $L^{(p,\infty)}$ is called the Marcinkiewicz space or Weak-L^p. We will also make use of PM^a spaces

$$PM^a := \left\{ v \in S' : \hat{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ \|v\|_{PM^a} \equiv \operatorname{ess\,sup}_{\xi\in\mathbb{R}^n} |\xi|^a |\hat{v}(\xi)| < \infty \right\},$$

where $a \ge 0$. For more details on these functional spaces see, for instance, [1,7,12,14].

2. An equation in Banach spaces

The following theorem generalizes the results in [3] (Lemmas 3 and 8) and in [12] (Theorem 13.2). The proof is based on the standard Picard iteration technique together with the Banach fixed point theorem. In what follows, X denotes a Banach space with norm $\|\cdot\|$. For X and Y Banach spaces, we refer to the Banach space $X \cap Y$ in the usual context, that is, when X and Y are embedded in the same topological vector space, and $X \cap Y$ is Banach with respect to the sum or maximum norm.

Theorem 2.1. *Let $m \in \mathbb{N}$ and, for each $1 < i < m$, let $\rho_i \in (1, \infty)$ and $K_i > 0$. Suppose $T : X \to X$ to be a linear continuous map satisfying*

$$\|T(x) - T(z)\| \le \tau\|x - z\|, \quad \forall x, z \in X, \tag{2}$$

for a fixed $\tau < 1$. For each $1 < i < m$, let $S_i : X \to X$ be a map satisfying

$$\|S_i(x) - S_i(z)\| \le K_i\|x - z\|\left(\|x\|^{\rho_i-1} + \|z\|^{\rho_i-1}\right), \tag{3}$$

for all $x, z \in X$. Let $R > 0$ be the least positive root of equation $\sum_{i=1}^m \frac{2^{\rho_i} K_i}{(1-\tau)^{\rho_i-1}} R^{\rho_i-1} + \tau - 1 = 0$. We have:

1. *(Existence and Uniqueness). Given $0 < \varepsilon < R$, if $y \in X$ is such that $\|y\| \le \varepsilon$, then there exists a solution $x \in X$ of the equation*

$$x = y + \sum_{i=1}^m S_i(x) + T(x), \tag{4}$$

satisfying $\|x\| \le \frac{2\varepsilon}{1-\tau}$. The solution x is unique in the ball $B_{\frac{2\varepsilon}{1-\tau}}$.