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Positive solutions for second-order four-point boundary value problems with alternating coefficient

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Abstract

In this paper, we investigate the existence of positive solutions for a class of nonlinear second-order four-point boundary value problems with alternating coefficient. Our approach relies on the Krasnosel'skii fixed point theorem. The result of this paper is new and extent the previously known result.

MSC: 34B10; 34B18

Keywords: Positive solution; Four-point boundary value problems; Coefficient that changes sign; Cone

1. Introduction

It is well known that the second-order boundary value problems play a very important role in both theories and applications. Recently, the existence of positive solutions for second-order three-point or four-point boundary value problems has been studied by many authors using various methods. See, for example, [1,2,4–7,9–13].

Motivated by [1,3,9], in this paper, we study the existence of positive solutions for second-order four-point boundary value problems as follows

$$\begin{cases} u''(t) + \lambda a(t) f(u(t)) = 0, & t \in (0, 1), \\ u(0) = \alpha u(\xi), & u(1) = \beta u(\eta), \end{cases} \tag{1.1}$$

where λ is a positive real parameter, $0 < \xi < \eta < 1$, $0 \le \alpha, \beta < 1$, and f, a satisfying

(B_1) $f : [0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing;

(B_2) $a : [0, 1] \to \mathbb{R}$ is continuous and such that $a(t) \le 0$, $t \in [0, \xi]$; $a(t) \ge 0$, $t \in [\xi, \eta]$; $a(t) \le 0$, $t \in [\eta, 1]$. Moreover, $a(t)$ does not vanish identically on any subinterval of $[0, 1]$;

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(B_3) There exists constant τ_1 with $\xi < \tau_1 < \min\{\eta - \alpha(\eta-\xi), \beta\eta + (1-\beta)\xi\}$, such that

$$h(t) = \delta_1 a^+(\tau_1 - \delta_1 t) - \frac{1}{\mu_1} a^-(t) \geq 0, \quad t \in [0, \xi],$$

where $a^+(t) = \max\{a(t), 0\}$, $a^-(t) = -\min\{a(t), 0\}$, $\mu_1 = \min\{\alpha, \beta\}\frac{1-\eta}{1-\beta\eta}$ and $\delta_1 = \frac{\tau_1 - \xi}{\xi}$;

(B_4) There exists constant τ_2 with $\beta\eta + (1-\beta)\xi < \tau_2 < \eta$, such that

$$g(t) = \delta_2 a^+(\eta - \delta_2 t) - \frac{1}{\mu_2} a^-(\eta + t) \geq 0, \quad t \in [0, 1-\eta],$$

where $\mu_2 = \min\{\alpha, \beta\}\frac{\xi}{1-\alpha+\alpha\xi}$ and $\delta_2 = \frac{\eta - \tau_2}{1-\eta}$.

The problem (1.1) has been studied previously when $a(t)$ is nonnegative (see [1,2]). In [9], Liu used Krasnosel'skii fixed point theorem to investigate the existence of at least one positive solution of the problem (1.1) with alternating coefficient when $\alpha = 0$. However, to the author's knowledge, no one has studied the existence of positive solutions for second-order four-point boundary value problems (1.1) with alternating coefficient. The aim of this paper is to fill the gap in the relevant literature. By using the Krasnosel'skii fixed point theorem, we will establish some simple criteria for the existence of positive solutions of BVP (1.1) under some conditions concerning the function a that is sign-changing on [0, 1].

Remark 1.1. We point out that conditions (B_3) and (B_4) are reasonable. For example, we take $\xi = \frac{1}{5}$, $\eta = \frac{1}{2}$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{3}$, $\tau_1 = \frac{1}{4}$, $\tau_2 = \frac{2}{5}$, and

$$a(t) = \begin{cases} \dfrac{15}{128}\left(t - \dfrac{1}{5}\right), & t \in \left[0, \dfrac{1}{5}\right], \\ 50\left(t - \dfrac{1}{5}\right)\left(\dfrac{1}{2} - t\right), & t \in \left[\dfrac{1}{5}, \dfrac{1}{2}\right], \\ \dfrac{1}{40}\left(\dfrac{1}{2} - t\right), & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Then, it is easy to check that $\frac{1}{5} = \xi < \tau_1 < \min\{\eta - \alpha(\eta-\xi), \beta\eta + (1-\beta)\xi\} = \frac{3}{10}$, $\frac{3}{10} = \beta\eta + (1-\beta)\xi < \tau_2 < \eta = \frac{1}{2}$, $\mu_1 = \frac{3}{20}$, $\mu_2 = \frac{1}{16}$, $\delta_1 = \frac{1}{4}$ and $\delta_2 = \frac{1}{5}$. Moreover, for $t \in [0, \frac{1}{5}]$, we have

$$\begin{aligned} h(t) &= \delta_1 a^+(\tau_1 - \delta_1 t) - \frac{1}{\mu_1} a^-(t) = \frac{1}{4} a^+\left(\frac{1}{4} - \frac{1}{4}t\right) - \frac{20}{3} a^-(t) \\ &= \frac{25}{32} t \left(\frac{1}{5} - t\right) \geq 0. \end{aligned}$$

For $t \in [0, \frac{1}{2}]$, one has

$$\begin{aligned} g(t) &= \delta_2 a^+(\eta - \delta_2 t) - \frac{1}{\mu_2} a^-(\eta + t) = \frac{1}{5} a^+\left(\frac{1}{2} - \frac{t}{5}\right) - 16 a^-\left(\frac{1}{2} + t\right) \\ &= \frac{t}{5}(1 - 2t) \geq 0. \end{aligned}$$

Thus, conditions (B_3) and (B_4) hold, and condition (B_2) also holds.

2. Some lemmas

Lemma 2.1 ([1]). *If* $M := \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) \neq 0$, *Green's function for the boundary value problem*

$$\begin{cases} -u''(t) = 0, & t \in (0, 1), \\ u(0) = \alpha u(\xi), & u(1) = \beta u(\eta), \end{cases} \tag{2.1}$$