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Fractional functional differential inclusions with finite delay

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Abstract

In this paper, we present fractional versions of the Filippov theorem and the Filippov–Wazewski theorem, as well as an existence result, compactness of the solution set and Hausdorff continuity of operator solutions for functional differential inclusions with fractional order,

$$D^\alpha y(t) \in F(t, y_t), \quad \text{a.e. } t \in [0, b],\ 0 < \alpha < 1,$$
$$y(t) = \phi(t), \quad t \in [-r, 0],$$

where $J = [0, b]$, D^α is the standard Riemann–Liouville fractional derivative, and F is a set-valued map.

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1. Introduction

This paper deals with the existence of solutions, for initial value problems (IVP for short), for fractional differential inclusions with infinite delay

$$D^\alpha y(t) \in F(t, y_t), \quad \text{a.e. } t \in J := [0, b],\ 0 < \alpha < 1, \tag{1}$$
$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{2}$$

where D^α is the standard Riemann-Liouville fractional derivative, $F : J \times C([-r, 0], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $\phi \in C([-r, 0], \mathbb{R})$, and $\phi(0) = 0$.

For any function y defined on $[-r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t.

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Differential equations with fractional order have been recently proved to be valuable tools in the modeling of many physical phenomena [10,20,21,36,37,3,9,18,31,32,41,43].

The arbitrary (fractional) order integral (Riemann–Liouville) operator is a singular integral operator, and for details on the arbitrary fractional order differential operator, see the references [14,15,17]. There has been a significant development in fractional differential equations in recent years; see the monographs of Miller and Ross [40], Samko et al. [45] and the papers of Diethelm et al. [10,12,13], El-Sayed [16], Mainardi [36], Momani and Hadid [38], Momai et al. [39], Podlubny et al. [44], and Yu and Gao [48,49].

Very recently, some basic theory for initial value problems of fractional differential equations involving the Riemann–Liouville differential operator was discussed by Benchohra et al. [5] and Lakshmikantham and Vastala [33–35].

El-Sayed and Ibrahim [17] and Benchohra et al. [6] initiated the study of fractional multivalued differential inclusions.

In the case where $\alpha \in (1, 2]$, existence results for fractional boundary value problems and a relaxation theorem were studied by Ouahab [42].

Our goal in this paper is to complement and extend some of the above results by giving some existence results and a relaxation theorem for Problem (1) and (2). Also,we prove that the set of solutions is compact.

2. Preliminaries

Throughout this paper we will use the following notations: $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space; see [30].

Definition 2.1. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x, \ y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For more details on multivalued maps we refer to the books of Aubin and Celina [1], Aubin and Frankowska [2], Deimling [11], Gorniewicz [22], Hu and Papageorgiou [23,24], Kamenskii et al., [29], Smirnov [46] and Tolstonogov [47].

We consider $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, and let $C^0(\mathbb{R}^+)$ denote the space of all continuous functions on $\mathbb{R}^+$. Consider also the space $C^0(\mathbb{R}_0^+)$ of all continuous real functions on

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\},$$

which later, by a slight abuse of the notation, will also be identified with the class of all $f \in C^0(\mathbb{R}^+)$ such that $\lim_{t\to 0^+} f(t) = f(0^+) \in \mathbb{R}$.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Definition 2.2. The fractional primitive of order $\alpha > 0$ of a function $h : \mathbb{R}^+ \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s,$$